37. A Note on the Entropy for Operator Algebras

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Recently, I. E. Segal [9] established the notion of the entropy of states of semi-finite von Neumann algebras. Segal's entropy contains the cases of the information theory, e.g. A. I. Khinchin [5], and the quantum statistical mechanics due to J. von Neumann [8]. The purpose of the present note is to discover the background of Segal's definition basing on a study of the so-called convex operator functions due to originally C. Loewner and extensively J. Bendat and S. Sherman [1].

1. A real-valued continuous function f defined on an interval I will be called *operator-convex* in the sense of Loewner-Bendat-Sherman provided that

$$f(\alpha a+\beta b)\leqq\alpha f(a)+\beta f(b), \tag{1}$$

for any hermitean operators a and b having their spectra in I, and for any non-negative real numbers α and β with $\alpha+\beta=1$. According to a theorem of Bendat-Sherman [1; Theorem 3.5], an analytic function,

$$f(\lambda)=\sum_{i=2}^{\infty}\gamma_i\lambda^i, \tag{2}$$

with the convergence radius R, is operator-convex for $|\lambda|<R$ if and only if

$$\sum_{i,k=0}^{n}\frac{f^{(i+k+2)}(0)}{(i+k+2)!}\alpha_i\alpha_k\geqq 0, \tag{3}$$

for any sequence of real numbers α_i and for all n.

LEMMA 1. *$\lambda\log(1+\lambda)$ is operator-convex for $|\lambda|<1$.*

Proof. Put $f(\lambda)=\lambda\log(1+\lambda)$. Clearly f satisfies (2) for $R=1$. Calculating, for $k=2,3,\cdots$,

$$f^{(k)}(\lambda)=(-1)^k[(k-2)!\,(1+\lambda)^{-(k-1)}+(k-1)!\,(1+\lambda)^{-k}].$$

Putting $\lambda=0$, one has $f^{(k)}(0)=(-1)^k(k-2)!\,k$ for $k=2,3,\cdots$. Applying (3), one has, for any real numbers α_i,

$$\sum_{i,k=0}^{n}\frac{f^{(i+k+2)}(0)}{(i+k+2)!}\alpha_i\alpha_k=\sum_{i,k=0}^{n}(-1)^{i+k}\frac{(i+k)!\,(i+k+2)}{(i+k+2)!}\alpha_i\alpha_k$$

$$=\sum_{i,k=0}^{n}(-1)^{i+k}\frac{\alpha_i\alpha_k}{i+k+1}.$$

Replacing $(-1)^i\alpha_i$ by α_i, it is non-negative, since the matrix,

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