29. *On the Curvature of Parametric Curves*

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(Comm. by Z. SUETUNA, M.J.A., March 13, 1961)

1. **Introduction.** In classical differential geometry the treatment of the curvature of parametric curves is restricted to the case in which the curves are at least twice continuously differentiable. As a contrast to this, on the other hand, we have in real function theory the fundamental theorem of Lebesgue, according to which a function of a real variable is almost everywhere derivable provided it is of bounded variation. And yet the two quantities, curvature and derivative, may be thought to belong by origin to a common mathematical category, in the sense that they both are outcomes of the same process of differentiation, applied once or twice according to the cases. Reflecting upon this fact we are led to surmise that a theory of curvature might be constructed under more general assumptions on the curves than usual. It is the object of the present note to show that such a theory is actually possible. The tools requisite thereto are already obtained in our recent papers [1] to [4].

2. **Bend of parametric curves.** In what follows the term *interval*, by itself, will always mean a linear interval in its widest sense, i.e. any connected infinite set of real numbers. As usual the prepositive epithets *closed* and *open* for intervals will only be used in connection with finite (that is, bounded) intervals, while we shall term *endless* any interval which is an open set.

Consider a fixed Euclidean space $\boldsymbol{R}^m$ of any dimension $m \geqq 2$. The points of $\boldsymbol{R}^m$ will be regarded as vectors whenever convenient. We shall denote by $p \circ q$ the angle made by any pair of nonvanishing vectors p, q of $\boldsymbol{R}^m$ and contained in the closed interval $[0, \pi]$. By a *parametric curve*, or simply *curve*, in $\boldsymbol{R}^m$ we shall understand an arbitrary mapping of the real line $\boldsymbol{R}$ into the space $\boldsymbol{R}^m$. A curve will be called to be *light*, if it is constant on no intervals.

The letter φ will stand in the present and the next section for a given light curve. We call *bend* of φ on an interval I and denote by $\Omega(\varphi, I)$, the quantity defined as follows. Let Δ be any finite, nonoverlapping sequence of closed intervals $J_1, \cdots, J_n$ $(n \geqq 2)$ situated in I. We require further that none of the increments $\varphi(J_i)$ of φ over them vanish and that these intervals are arranged in Δ in the same order in which they appear in the real line $\boldsymbol{R}$ (so that J_i lies in $\boldsymbol{R}$ on the left of J_{i+1} for $i=1, 2, \cdots, n-1$). Plainly the former requirement