28. *On Invariant Groups of m-forms*

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(Comm. by K. SHODA, M.J.A., March 13, 1961)

1. The algebraic dimensions of invariant groups (orthogonal groups) of quadratic forms are uniquely determined by the number of their variables. But, those of the invariant groups of m-forms ($m\geq 3$) are not uniquely determined with the number of their variables.

We shall determine two types of invariant groups of m-forms, the one's algebraic dimension is zero[1] and the other's is not zero.

2. Let k be a field of characteristic 0, and V be an n-dimensional vector space over k. We shall say, $F(X)$ is an m-form defined on V, if there exists a symmetric m-linear form $f(X^{(1)}, X^{(2)},\cdots, X^{(m)})$ defined on the direct product of m-copies of V, such that $F(X)=f(X, X,\cdots, X)$. Every homogeneous polynomial with n-variables and of degree m is an m-form.

For an m-form $F(X)$, the F-radical N_F of V, is a subspace of V consisting of all vectors X, which satisfy the equation $f(X^{(1)}, X^{(2)}, \cdots, X^{(m-1)}, X)=0$, for any vectors $X,^{(1)} X^{(2)},\cdots, X^{(m-1)}$, in V.

If $N_F\neq 0$ then we shall say that $F(X)$ is non-degenerate, and if $N_F=0$, degenerate. When $F(X)$ is degenerate, there exists the non-degenerate m-form defined on V/N_F, induced by $F(X)$.

We shall use $E(V)$ to denote the ring of k-linear endomorphisms of V, and $G(F)$, the subset of $E(V)$ consisting of all endomorphisms A which leave $F(X)$ invariant, i.e. $F(X)=F(X\cdot A)$.

Proposition 1. *If $F(X)$ is non-degenerate, $G(F)$ is a group.*

Proof. We have to show that every endomorphism A, belonging to $G(F)$ is an automorphism of V.

If A is not an automorphism, there exists a non-zero vector X in V, which satisfies $X\cdot A=0$. Then

$$f(X^{(1)}, X^{(2)},\cdots, X^{(m-1)}, X)=f(X^{(1)}\cdot A, X^{(2)}\cdot A,\cdots, X^{(m-1)}\cdot A, X\cdot A)=0$$

for any vectors $X^{(1)}, X^{(2)},\cdots, X^{(m-1)}$. This implies that N_F contains non-zero vector X. And this contradiction shows that A is an automorphism.

If $F(X)=\sum_{i=1}^{n} a_i x_i^m$, then we shall say that $F(X)$ is a diagonal form.

Proposition 2. *When $F(X)$ is a diagonal form, then $F(X)$ is non-*

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1) The algebraic dimensions of the invariant groups of m-forms are zero, if and only if the group is a finite group (cf. C. Chevalley: Théorie des Groupes de Lie, 2, Hermann, Paris (1951)).