# 51. The Local Structure of an Orbit of a Transformation Group

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A topological group $G$ is said to act on a topological space $M$ when the following conditions are satisfied:

(1) the elements of $G$ are homeomorphisms of $M$ onto itself,

(2) the mapping $(g, x)\rightarrow g(x)$ of $G\times M$ onto $M$ is continuous,

(3) $g_1(g_2(x))=(g_1g_2)(x)$ for every $x\in M$ and $g_1, g_2\in G$.

In the following $G$ will denote a locally compact group satisfying the second axiom of countability, $G_0$ the identity component of $G$, and $M$ a Hausdorff space throughout this note.

Montgomery and Zippin [7] proved that if $G$ is a compact group acting on a $k$-dimensional orbit $M$, then $M$ is locally the topological product of a $k$-cell by a compact zero dimensional set. In the general case where $G$ is locally compact, as a counter example shows, the above fact is not true, but it holds if only the zero dimensional set is "closed in $M$" instead of "compact" (the main theorem). As a corollary of this fact it is proved that if $G$ acts transitively and effectively on a finite dimensional connected locally connected space $M$ then $G$ is a Lie group (Corollary 1). Moreover the assumption that $M$ is connected is redundant in this corollary when $G/G_0$ is compact or $G$ is abelian (Corollary 2).

As $G$ satisfies the second axiom of countability, all factor spaces and orbits of $G$ are separable metric, so that we can make free use of dimension theory (cf. [4]). For topological and group-theoretical terms, we follow the usage of Montgomery and Zippin [6].

The following Lemma 1 was proved by Montgomery and Zippin [7] when $G$ is compact. Using the structure theorem of locally compact groups (cf. [6], p. 175), their proof remains true as it is when $G$ is locally compact and $G/G_0$ is compact.

**Lemma 1** (Montgomery and Zippin [7]). *If $G/G_0$ is compact and $G$ acts on a finite dimensional orbit $G(x)$, then $G$ is effectively finite dimensional on $G(x)$. In fact, there must be a connected compact invariant subgroup $K$ which is idle on $G(x)$ and such that $G/K$ is finite dimensional.*

**Lemma 2.** *Let $G$ be a finite dimensional group, and $H$ a closed subgroup of $G$. Then there is such an arbitrarily small compact local cross section $W$ of cosets of $H$ as the form $LZ$, where $L$ is a compact local Lie subgroup of $G$ and $Z$ is a compact zero dimensional*