47. On the Definition of the Cross and Whitehead Products in the Axiomatic Homotopy Theory. I

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1. Introduction. The axiomatic characterization of homotopy theory has been established by several authors [3–5]. In [3], S. T. Hu proposed the problem to present a new definition of the Whitehead products so that they might be fixed into the axiomatic scheme without appeal to the geometric representation of the homotopy groups. In this paper, we shall give an answer of this problem. First, the cross products will be defined; this is accomplished by using the commutator maps in loop spaces (cf. [2] and [6]). Then the Whitehead products will be given by the well-known formula $[\alpha, \beta] = F_{\#}\partial(\alpha \times \beta)$ for $\alpha \in \pi_m(X, x_0)$, $\beta \in \pi_n(X, x_0)$, where F: $X \vee X \to X$ is the folding map (cf. [1] or [7]).

2. The cross and the Whitehead products. Let (X, A, x_0) be a triplet of topological spaces and let $\Omega(X; A, x_0)$ be a path space of X consisting of maps σ: $I \to X$ such that $\sigma(0) \in A$, $\sigma(1) = x_0$. A constant path σ: $I \to X$ such that $\sigma(t) = x_0$ for all $t \in I$ will be denoted by the same letter x_0 and is considered as a base point of $\Omega(X; A, x_0)$. If any confusion does not occur, we denote $\Omega(X; A, x_0)$ simply by $\Omega(X, A)$ and $\Omega(X, x_0)$ by ΩX. For a map f: $(X, A, x_0) \to (Y, B, y_0)$, we shall denote by Ωf: $\Omega(X; A, x_0) \to \Omega(Y; B, y_0)$ a map such that $(\Omega f)(x)(t) = f(x(t))$.

Let $H = \{\pi, \#, \partial\}$ be a given axiomatic homotopy theory (we shall denote the multiplication in homotopy groups by $+$). It is shown in [3] that there exists an isomorphism

$$\Omega\colon \pi_n(X, A, x_0) \approx \pi_{n-1}(\Omega(X, A), x_0), \qquad \text{for } n > 0.$$

Now, a cross product operation is a function defined for all pairs of homotopy groups $\pi_m(X, x_0)$ and $\pi_n(Y, y_0)$ of any two spaces (X, x_0) and (Y, y_0) and it is required to satisfy the following conditions.

Let $\alpha \in \pi_m(X, x_0)$, $\beta \in \pi_n(Y, y_0)$.

(a) In case $m = n = 0$. According to the definition of 0-dimensional homotopy groups, α and β are considered as path-components of X and Y, respectively. Let $x \in \alpha$, $y \in \beta$. A path component γ of $X \times Y$ modulo $X \vee Y$ $(= X \times y_0 \cup x_0 \times Y)$ containing (x, y) depends only on α and β. A cross product of α and β is defined by

$$\alpha \times \beta = \gamma \in \pi_0(X \times Y, X \vee Y, (x_0, y_0)).$$