43. Continuity Properties on the Retardation in the Theory of Difference-Differential Equations

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(Comm. by Z. SUETUNA, M.J.A., April 12, 1961)

Introduction. In their paper,*) Bellman and Cooke have discussed the uniform convergence of solutions of a certain type of difference-differential equations as the retardation approaches zero, for which they applied the successive approximation method as a tool.

In this paper, we shall prove that the uniform convergence of solutions of difference-differential equations as the retardation approaches zero is a natural consequence of the continuity of solutions with respect to the retardation. The equation to be discussed here is rather general than that in their paper cited above. Among the variables appearing in the sequel, t represents a scalar and x, f may be vectors real or complex. By $|x|$ we denote a norm of x.

1. Continuity properties of solutions. We shall consider a difference-differential equation

(1.1) $$x'(t)=f(t, x(t), x(t-h))$$

for $0\leqq t\leqq t_0$, where h is a positive constant. The initial conditions imposed on (1.1) are that

(1.2) $$x(t)=\phi(t) \ (-h\leqq t\leqq 0) \quad \text{and} \quad x(0)=x_0,$$

where $\phi(t)$ is a given function continuous for $-h\leqq t\leqq 0$, $\phi(0)=x_0$, and $|\phi(t)-x_0|\leqq a$. Then, it is well known that if the function $f(t, x, y)$ satisfies the following conditions:

(i) $f(t,x,y)$ is continuous and $|f(t,x,y)|\leqq M$ for

(1.3) $$0\leqq t\leqq t_0, \quad |x-x_0|\leqq a, \quad |y-x_0|\leqq a;$$

(ii) $f(t,x,y)$ satisfies the Lipschitz condition, that is,

$$|f(t, x_1, y_1)-f(t, x_2, y_2)|\leqq k_1|x_1-x_2|+k_2|y_1-y_2|$$

for any t, x_i, y_i $(i=1,2)$ in the domain (1.3), where k_1 and k_2 are constants, then the existence and uniqueness of continuous solutions of (1.1) under the initial conditions (1.2) are established for $0\leqq t\leqq t^*=\min(t_0, a/M)$.

Now, for any positive constants h_i $(i=1,2)$ not greater than h, we consider two equations

(1.4) $$x'(t)=f(t, x(t), x(t-h_i)) \quad (i=1,2)$$

*) R. Bellman and K. L. Cooke: On the limit of solutions of differential-difference equations as the retardation approaches zero, Proc. Nat. Acad. Sci., U.S.A., **45**, 1026-1028 (1959).