58. On Open Mappings. II

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(Comm. by K. KUNUGI, M.J.A., May 13, 1961)

Let X and Y be topological spaces and let f be a continuous mapping of X onto Y. f is said to be open if the image of every open subset of X is open in Y. A. H. Stone [9] has obtained conditions under which the image of an open continuous mapping of a metric space becomes metrizable. In this note, we shall obtain some results concerning the images of the open continuous mappings of metric spaces.

1. By the *open image*, we mean the image of an open continuous mapping. We begin with proving the following theorem.

Theorem 1. *If X is a T_1-space which satisfies the first countability axiom, then X is an open image of a metric space.*

Proof. Let $\{U_\alpha \mid \alpha\in\Omega\}$ be the open basis of X. For each point x of X, let $\{U_{\alpha_i} \mid i=1,2,\cdots;\ \alpha_i\in\Omega\}$ be an open neighborhood basis of x, then $\alpha=(\alpha_1,\alpha_2,\cdots)\in N(\Omega)$, where $N(\Omega)$ is the generalized Baire's zero-dimensional space*) introduced by K. Morita [4]. Now let A denote the set of all such α. If we define a mapping f of A into X by $f(\alpha)=x$, then it is evident that $f(A)=X$. We shall next prove that f is an open continuous mapping. Let V be any open neighborhood of x such that $f(\alpha)=x$, then, since $\{U_{\alpha_i} \mid i=1,2,\cdots\}$ is an open neighborhood basis of x, there exists a U_{α_k} such that $U_{\alpha_k}\subset V$. Then if $\rho(\alpha,\beta)<\frac{1}{k}$ where $\beta=(\beta_1,\beta_2,\cdots)\in A$, then $\alpha_i=\beta_i$ for $i\leqq k$ by the definition of the metric of $N(\Omega)$. Hence $f(\beta)\in\bigcap_{i=1}^{k}U_{\alpha_i}\subset U_{\alpha_k}\subset V$. Therefore f is continuous.

Now let $V\left(\alpha;\frac{1}{k}\right)=\left\{\beta \mid \rho(\alpha,\beta)<\frac{1}{k}\right\}$, then $f\left(V\left(\alpha;\frac{1}{k}\right)\right)=\bigcap_{i=1}^{k}U_{\alpha_i}$. In fact, since $f\left(V\left(\alpha;\frac{1}{k}\right)\right)\subset\bigcap_{i=1}^{k}U_{\alpha_i}$, it is sufficient to show that $f\left(V\left(\alpha;\frac{1}{k}\right)\right)\supset\bigcap_{i=1}^{k}U_{\alpha_i}$. For this purpose, let $y\in\bigcap_{i=1}^{k}U_{\alpha_i}$ and let $\{U_{\beta_j} \mid j=k+1, k+2,\cdots\}$ be an open neighborhood basis which is obtained by number-

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*) We define the metric ρ of $N(\Omega)=\{(\alpha_1,\alpha_2,\cdots) \mid \alpha_i\in\Omega, i=1,2,\cdots\}$ as follows: if $\alpha=(\alpha_1,\alpha_2,\cdots)$, $\beta=(\beta_1,\beta_2,\cdots)$, $\alpha_i=\beta_i$ for $i<n$, $\alpha_n\neq\beta_n$, then $\rho(\alpha,\beta)=\frac{1}{n}$. As is well known, $N(\Omega)$ is a 0-dimensional metric space and we call $N(\Omega)$ a generalized Baire's zero-dimensional space according to K. Morita.