73. A Generalization of the Heinz Inequality

By Tosio KATO
Department of Physics, University of Tokyo
(Comm. by K. KUNUGI, M.J.A., June 12, 1961)

The object of the present note is to generalize the Heinz inequality for selfadjoint operators to a wider class of accretive operators.

A linear operator A in a Hilbert space is said to be *accretive*[1] if $\mathrm{Re}(Au, u)\geq 0$ for all $u\in\mathfrak{D}[A]$ ($\mathfrak{D}[A]$ is the domain of A). If A is closed and maximal accretive, then A is densely defined, and the fractional powers A^h are defined for $0\leq h\leq 1$ and are again closed and maximal accretive.[2]

Our main result is given by

Theorem 1. *Let A, B be closed, maximal accretive operators in Hilbert spaces $\mathfrak{H}$, $\mathfrak{H}'$, respectively, and let T be a bounded linear operator*[3] *from $\mathfrak{H}$ to $\mathfrak{H}'$. If $T\mathfrak{D}[A]\subset\mathfrak{D}[B]$ and*

$$(1)\qquad \|BTu\|\leq M\|Au\|,\quad u\in\mathfrak{D}[A],$$

with a constant M, then we have $T\mathfrak{D}[A^h]\subset\mathfrak{D}[B^h]$ and

$$(2)\qquad \|B^hTu\|\leq e^{ch(1-h)}M^hN^{1-h}\|A^hu\|,\quad u\in\mathfrak{D}[A^h],$$

where $N=\|T\|$, $0\leq h\leq 1$ and c is an absolute constant. We can take $c=0$ if A, B are selfadjoint and nonnegative. In general we can take $c=\pi^2/2$, but we do not know whether this is the optimal value.

Remark. The value of c can be improved if A, B are themselves fractional powers of accretive operators. Suppose that there are closed, maximal accretive operators A_1, B_1 in $\mathfrak{H}$, $\mathfrak{H}'$, respectively, such that $A=A_1^s$, $B=B_1^t$ for some s, t, $0<s\leq 1$, $0<t\leq 1$. Then we can set $c=\pi^2(s^2+t^2)/4$. (The proof is not essentially different from the proof of Theorem 1 given below.) If, for example, A is nonnegative selfadjoint, we can make $s\to 0$ and set $c=\pi^2t^2/4$.

Corollary. *If A, B are closed, maximal accretive operators in $\mathfrak{H}$ such that $\mathfrak{D}[A]\subset\mathfrak{D}[B]$ and $\|Bu\|\leq\|Au\|$ for $u\in\mathfrak{D}[A]$, then $\mathfrak{D}[A^h]\subset\mathfrak{D}[B^h]$ and $\|B^hu\|\leq e^{ch(1-h)}\|A^hu\|$ for $u\in\mathfrak{D}[A^h]$, $0\leq h\leq 1$.*

Theorem 1 is equivalent to

Theorem 2. *Let A, B be as in Theorem 1, and let Q be a densely*

---

1) Then $-A$ is said to be *dissipative*. For the term "accretive", see K. O. Friedrichs: Symmetric positive linear differential equations, Comm. Pure Appl. Math., **11**, 333-418 (1958).

2) See T. Kato: Fractional powers of dissipative operators, J. Math. Soc. Japan, **13** (1961), in press. This paper will be quoted as (F) in the following.

3) A bounded linear operator is assumed to be defined everywhere in the domain space, unless otherwise stated explicitly.