72. Inverse Images of Closed Mappings. II

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In the following, we deal chiefly with the case when the inverse images of closed continuous mappings become normal.

Theorem 5. *Let $f(X)=Y$ be a closed continuous mapping of a topological space X onto a paracompact Hausdorff space Y. Then X is normal if and only if, for each point y of Y, any two disjoint closed subsets A, B of the inverse image $f^{-1}(y)$ can be separated by open sets of X, that is, there exist open sets G, H of X such that $G\supset A$, $H\supset B$ and $G\frown H=\phi$.*

Proof. The "only if" part is obvious. So that we shall prove the "if" part. Let A and B be two disjoint closed sets of X and let G be an open set of X. Then we can see that the set $\{y\mid f^{-1}(y)\frown A\subset G\}$ is an open set of Y. In fact, let y_0 be any point such that $f^{-1}(y_0)\frown A\subset G$ and let $V=Y-f(A\frown(X-G))$. Then, since f is a closed continuous mapping, V is an open set of Y and $y_0\in V$, $f^{-1}(V)\frown A\frown(X-G)=\phi$. Hence $f^{-1}(V)\frown A\subset G$. Therefore the set $\{y\mid f^{-1}(y)\frown A\subset G\}$ is an open set of Y. Now let $U_G=\{y\mid f^{-1}(y)\frown A\subset G, f^{-1}(y)\frown B\subset X-\bar{G}\}$, then U_G is an open set of Y. For any point y_0 of Y, $f^{-1}(y_0)\frown A$ and $f^{-1}(y_0)\frown B$ are disjoint closed sets of $f^{-1}(y_0)$. By assumption, there exist two open sets G_0, H_0 of X such that $f^{-1}(y_0)\frown A\subset G_0$, $f^{-1}(y_0)\frown B\subset H_0$ and $G_0\frown H_0=\phi$. Since $\bar{G}_0\frown H_0=\phi$, we get $H_0\subset X-\bar{G}_0$. Hence $y_0\in U_{G_0}$. Then we can see that the family of open sets $\{U_G\mid G$ ranges over all open sets of $X\}$ is an open covering of Y. Since Y is paracompact Hausdorff space, there exists a locally finite open covering $\{V_G\mid G\in\mathfrak{G}\}$ where $\mathfrak{G}$ is a family of open sets of X such that $\bar{V}_G\subset U_G$ for every $G\in\mathfrak{G}$. Let $H=\bigcup_{G\in\mathfrak{G}}(f^{-1}(V_G)\frown G)$, then H is an open set of X and $\{f^{-1}(V_G)\frown G\mid G\in\mathfrak{G}\}$ is locally finite. Hence $\bar{H}=\bigcup_{G\in\mathfrak{G}}(\overline{f^{-1}(V_G)\frown G})\subset\bigcup_{G\in\mathfrak{G}}(f^{-1}(\bar{V}_G)\frown\bar{G})$. On the other hand, since $f^{-1}(V_G)\frown A\subset f^{-1}(U_G)\frown A\subset G$, we get $f^{-1}(V_G)\frown A\subset f^{-1}(V_G)\frown G\subset H$. Since $\{f^{-1}(V_G)\mid G\in\mathfrak{G}\}$ covers X, we get $A\subset H$. On the other hand, $f^{-1}(\bar{V}_G)\frown B\frown\bar{G}\subset f^{-1}(U_G)\frown B\frown\bar{G}\subset(X-\bar{G})\frown\bar{G}=\phi$. Then $B\frown\bar{H}=\phi$. Hence we have an open set $X-\bar{H}$ which contains B. Therefore A and B are separated by open sets H and $X-\bar{H}$, and so that X is normal. This completes the proof.