71. *Inverse Images of Closed Mappings. I*

By Sitiro HANAI
Osaka University of Liberal Arts and Education
(Comm. by K. KUNUGI, M.J.A., June 12, 1961)

Let f be a closed continuous mapping of a topological space X onto a topological space Y. It is well known that if Y is paracompact and $f^{-1}(y)$ is compact for each point y of Y, then X is paracompact [2, 3]. It is interesting to know under what conditions the topological properties of Y may be preserved by the inverse mapping f^{-1}. H. Tamano [7] has recently obtained the necessary and sufficient condition that the inverse image space $X=f^{-1}(Y)$ be normal where X and Y are completely regular T_1-spaces and Y is paracompact.

In this note, we shall investigate the compactness of the inverse image space $X=f^{-1}(Y)$ under the closed continuous mapping f.

In the first place, let us quickly recall some definitions which were introduced by K. Morita [5]. For any infinite cardinal number $\mathfrak{m}$, a topological space X is said to be $\mathfrak{m}$-paracompact if any open covering of X with power $\leqq \mathfrak{m}$ (i.e. consisting of at most $\mathfrak{m}$ sets) admits a locally finite open covering as its refinement. A topological space X is called $\mathfrak{m}$-compact if every open covering of power $\leqq \mathfrak{m}$ has a finite subcovering.

Theorem 1. *If f is a closed continuous mapping of a topological space X onto an $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact) topological space Y such that the inverse image $f^{-1}(y)$ is $\mathfrak{m}$-compact for every point y of Y, then X is $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact).*

Proof. Let $\mathfrak{U}=\{U_\lambda \mid \lambda\in\Lambda\}$, $|\Lambda|\leqq\mathfrak{m}$ be an open covering of X where $|\Lambda|$ denotes the power of Λ. And let Γ be the family of all finite subsets γ of Λ, then $|\Gamma|\leqq\mathfrak{m}$. Since $f^{-1}(y)$ is $\mathfrak{m}$-compact for every point y of Y, there exists a finite subset γ of Λ such that $f^{-1}(y)\subset\bigcup_{\lambda\in\gamma}U_\lambda$. Let $V_\gamma=Y-f(X-\bigcup_{\lambda\in\gamma}U_\lambda)$, then V_γ is open by the closedness of f and $y\in V_\gamma$ and $f^{-1}(V_\gamma)\subset\bigcup_{\lambda\in\gamma}U_\lambda$. Therefore $\mathfrak{V}=\{V_\gamma \mid \gamma\in\Gamma\}$ is an open covering of Y with power $\leqq\mathfrak{m}$. If Y is $\mathfrak{m}$-paracompact ($\mathfrak{m}$-compact), then there exists a locally finite (finite) open refinement $\{W_\delta \mid \delta\in\Delta\}$ of $\mathfrak{V}$. Since, for each δ there exists a $\gamma_\delta\in\Gamma$ such that $f^{-1}(W_\delta)\subset f^{-1}(V_{\gamma_\delta})\subset\bigcup_{\lambda\in\gamma_\delta}U_\lambda$, $\{f^{-1}(W_\delta)\frown U_\gamma \mid \delta\in\Delta, \lambda\in\gamma_\delta\}$ is a locally finite (finite) open refinement of $\mathfrak{U}$. Thus we get the theorem. From Theorem 1, we have the following corollaries (see [2, 3, 1]).

Corollary 1.1. *If f is a closed continuous mapping of a topological space X onto a paracompact topological space Y such that*