70. Remarks on Knots with Two Bridges

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§1. Introduction. In 1954, H. Schubert introduced the new numerical knot invariant, called the bridge number of the knot [6]. Then he completely classified the knots with two bridges [7]. He assigned two integers α and β to a knot k with two bridges. α is called a *torsion*, which is the same as the well-known second torsion number of k, and the other β is called "Kreuzungsklasse", whose new interpretation will be given in §2 in this note. As indicated by Schubert, the pair (α, β) will be called the *normal form* of k, where $\alpha>|\beta|>0$. After §3 the non-cyclic covering space $\mathfrak{F}$ unbranched along k will be considered following Bankwitz and Schumann [1]. Their discussion indicating that $\mathfrak{F}$ characterizes the knot plays an important role in classifying two knots of the same Alexander polynomial, as has been shown in their paper [1]. In §4 it will be shown that the Alexander polynomial over the Betti group of $\mathfrak{F}$ can be found based on the results in §3 following [3, III].

§2. Group presentation. Let k be a knot with two bridges of the normal form (α, β) and let K be its bridged projection. Let G be the knot group of k. The presentation of G will now be given based on K. K has $4p$ double points in which $2p$ double points lie in AB and the others lie in CD, where AB, CD are the *bridges*. These $4p$ double points will be named $X_1, X_2, \cdots, X_{2p}$ on AB, and $Y_1, Y_2, \cdots, Y_{2p}$ on CD in order of the direction of K starting at A. Then the over-presentation of G will be given by $G=(a, b\colon R, S)$, where $R=LaL^{-1}b^{-1}$, $S=MbM^{-1}a^{-1}$, $L=a^{\varepsilon_1}b^{\eta_1}a^{\varepsilon_2}b^{\eta_2}\cdots a^{\varepsilon_p}b^{\eta_p}$, $\varepsilon_i, \eta_j=+1$ or -1 for all i, j, and M is an element of G of the same type as L (cf. [4]), i.e. G is a group generated by two generators a, b and has two defining relations $R=1, S=1$. Since one of R, S is implied by the other, G can be considered as the group of a single defining relation R. And $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_p$ are defined as 1 or -1 depending on whether AB overpasses at $Y_1, Y_2, \cdots, Y_p$ from left to right or from right to left, and $\eta_1, \eta_2, \cdots, \eta_p$ are defined similarly. Thus it follows that

(2.1) G has a presentation as follows:

$$G=(a, b\colon R), \text{ where } R=LaL^{-1}b^{-1}.$$

In connection with this presentation, it follows that

(2.2) $2p+1$ equals α.