66. On Some Properties of Fractional Powers of Linear Operators

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A class of linear operators in a Banach space X is considered in a note by T. Kato.[1] A linear operator A in X is said to be of type (ω, M), if A is densely defined and closed, the resolvent set of $-A$ contains the open sector $|\arg\lambda|<\pi-\omega$, $0<\omega<\pi$, and $\lambda(\lambda+A)^{-1}$ is uniformly bounded in each smaller sector $|\arg\lambda|<\pi-\omega-\varepsilon$, $\varepsilon>0$, in particular $\lambda\,\|(\lambda+A)^{-1}\|\leq M$, $\lambda>0$. The fractional power A^α, $0<\alpha<1$, of A is defined by Kato through

$$(\lambda+A^\alpha)^{-1}=\frac{\sin\pi\alpha}{\pi}\int_0^\infty\frac{\mu^\alpha}{\lambda^2+2\lambda\mu^\alpha\cos\pi\alpha+\mu^{2\alpha}}(\mu+A)^{-1}d\mu,$$

where λ is in the sector $|\arg\lambda|<(1-\alpha)\pi$, and is shown to be of type $(\alpha\omega, M)$.

K. Yosida[2] gave an example showing that $(A^2)^{1/2}\neq A$ where $-A$ and $-A^2$ are infinitesimal generators of strongly continuous semi-groups. In this paper we shall prove, however, that $(A^\alpha)^\beta=A^{\alpha\beta}$, $0<\alpha$, $\beta<1$. We shall also prove that the semi-group $\{\exp(-tA^\alpha)\}$ generated by $-A^\alpha$ is continuous with respect to α in the uniform operator topology. This result overlaps with A. V. Balakrishnan's result[3] which says that $A^\alpha x$ is, for $x\in\mathfrak{D}(A)$, left-continuous at $\alpha=1$.

Theorem 1. Let A be of type (ω, M), then

$$(A^\alpha)^\beta=A^{\alpha\beta},\qquad 0<\alpha,\ \beta<1.$$

Proof. For any μ in the sector $|\arg\mu|<(1-\beta)\pi$

$$(\mu+(A^\alpha)^\beta)^{-1}=\frac{1}{(2\pi i)^2}\int_0^\infty\left(\frac{1}{\mu+\lambda^\beta e^{-i\pi\beta}}-\frac{1}{\mu+\lambda^\beta e^{i\pi\beta}}\right)d\lambda \int_0^\infty\left(\frac{1}{\lambda+\zeta^\alpha e^{-i\pi\alpha}}-\frac{1}{\lambda+\zeta^\alpha e^{i\pi\alpha}}\right)(\zeta+A)^{-1}d\zeta. \tag{1}$$

The double integral being absolutely convergent, we may interchange the order of the integration. Since we obtain

$$\frac{1}{2\pi i}\int_0^\infty\left(\frac{1}{\mu+\lambda^\beta e^{-i\pi\beta}}-\frac{1}{\mu+\lambda^\beta e^{i\pi\beta}}\right)\left(\frac{1}{\lambda+\zeta^\alpha e^{-i\pi\alpha}}-\frac{1}{\lambda+\zeta^\alpha e^{i\pi\alpha}}\right)d\lambda$$

1) T. Kato: Note on fractional powers of linear operators, Proc. Japan Acad., **36**, 94-96 (1960).

2) K. Yosida: Fractional powers of infinitesimal generators and the analyticity of the semi-groups generated by them, Proc. Japan Acad., **36**, 86-89 (1960).

3) A. V. Balakrishnan: Fractional powers of closed operators and the semi-groups generated by them, Pacific J. Math., **10**, 419-437 (1960).