65. On Neumann Problem for Laplace-Beltrami Operators

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§1. **Introduction.** In this paper, we deal with the second boundary value problem (Neumann problem) in compact subdomains of Riemannian spaces. Let $\boldsymbol{M}$ be an m-dimensional orientable Riemannian manifold of class C^3 with metric tensor $\| g_{ij}(x) \|$, and let D be a subdomain of $\boldsymbol{M}$ whose closure $\overline{D}$ is compact and whose boundary S consists of a finite number of $(m-1)$-dimensional hypersurfaces of class C^3. We denote by Λ the Laplace-Beltrami operator with respect to $\| g_{ij}(x) \|$:

$$\Lambda u(x)=\frac{1}{\sqrt{g(x)}}\frac{\partial}{\partial x^i}\left[\sqrt{g(x)}\, g^{ij}(x)\frac{\partial u(x)}{\partial x^j}\right] \quad \text{for} \quad u\in C^2(D)$$

where $\| g^{ij}(x) \| = \| g_{ij}(x) \|^{-1}$ and $g(x)=\det \| g_{ij}(x) \|$, and by $\dfrac{\partial}{\partial \boldsymbol{n}}$ the outer normal derivative at any point on the boundary S of D.

Consider the second boundary value problem in D associated with Λ:

$$\Lambda u=f \text{ in } D,\ \frac{\partial u}{\partial \boldsymbol{n}}=\varphi \text{ on } S, \tag{1.1}$$

where f and φ are given functions defined in D and on S respectively. The fundamental solution $U(t, x, y)$ of the initial-boundary value problem of the parabolic equation:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\Lambda u+f & (t>0,\ x\in D) \\ u|_{t=0}=u_0,\ \dfrac{\partial u}{\partial \boldsymbol{n}}=\varphi & (\text{on } S) \end{cases} \tag{1.2}$$

is given in [2] (see also [1]). We shall show that the kernel function $K(x, y)$ of the boundary value problem (1.1) is given by

$$K(x, y)=\int_0^\infty \{U(t, x, y)-|D|^{-1}\}\, dt \quad \text{whenever } x\neq y \tag{1.3}$$

where $|D|$ denotes the volume of D. Corresponding results in the case of Dirichlet problem, or in the case where Λ in (1.1) is replaced by $\Lambda-c(x)$ (here $c(x)$ is non-negative and not identically zero), are contained in [2; §10]; in these cases, the term $-|D|^{-1}$ in (1.3) should be omitted.

§2. **Main results.** In order that the boundary value problem (1.1) has a solution, the following condition is necessary:

$$\int_D f(x)\, dx=\int_S \varphi(x)\, dS_x \tag{2.1}$$