109. On a Theorem of Levine

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(Comm. by K. KUNUGI, M.J.A., Oct. 12, 1961)

1. Following after the notation of Terasaka, let a and i be the closure and interior operations on a topological space E respectively:

1) $A^{aa}=A^{a}$, 1′) $A^{ii}=A^{i}$,
2) $(A\smile B)^{a}=A^{a}\smile B^{a}$, 2′) $(A\frown B)^{i}=A^{i}\frown B^{i}$,
3) $A\leqq A^{a}$, 3′) $A\geqq A^{i}$,
4) $O^{a}=O$, 4′) $E^{i}=E$,

where O is the void set. It is well-known that they are related mutually by $i=cac$, where c is the complementation.

Very recently, N. Levine [2] proved the following interesting theorem:

THEOREM 1. *A subset A of E satisfies*

$$(1)\qquad A^{ai}=A^{ia},$$

if and only if there are a clopen set H and a nondense set P such that

$$(2)\qquad A=(H-P)\smile(P-H):$$

In short, A satisfies (1) *if and only if A is congruent to a clopen set H modulo nondense sets.*

Levine proved the theorem for T_1-spaces. However, the theorem is valid for closure algebras with a few modifications, which will be shown in §2. The remaining part of the proof of the theorem which is contained in §§2–3 is essentially same as that of Levine.

It will be interesting to observe that Levine's theorem has an application which characterizes the Borel sets of a hyperstonean space in terms of the closure and interior operations.

2. The following two identities guarantee that A^{c} *and* A^{a} *satisfy* (1) *whenever A satisfies* (1):

$$A^{cai}=A^{cacac}=A^{aic}=A^{ccacacc}=A^{cia},$$

and

$$A^{aia}=A^{ccacaca}=A^{ciaca}=A^{caica}=A^{cacacca}=A^{caca}=A^{ia}=A^{ai}=A^{aai}.$$

Consequently, A^{i} satisfies (1) if A satisfies (1), since $i=cac$.

It is clear that *a nondense set P and a clopen set H satisfy* (1), since $P^{ai}=O=P^{ia}$ and $H^{ai}=H=H^{ia}$.

It is also true that $H-P$ *satisfies* (1) *for clopen H and nondense P*: If $H>(H-P)^{ia}=(H\frown P^{ci})^{a}=(H\frown P^{ac})^{a}$, then

$$E=H\smile H^{c}>[(H\frown P^{ac})^{a}\smile(H^{c}\frown P^{ac})^{a}]=[(H\smile H^{c})\frown P^{ac}]^{a}=P^{aca}=E$$

shows a contradiction, whence $H=(H-P)^{ia}$. On the other hand,