105. On the Spectra of Some Non-linear Operators

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Let R be a real Banach space, K be a completely continuous, linear operator defined on R into itself and $\mathfrak{f}$ be a (in general, non-linear) continuous operator defined on R into itself.

In this note, we will study the properties of proper values and proper vectors of the operator $H=K\mathfrak{f}$. The integral operators of Hammerstein type are of this type.[1]

We denote by $S(H)$ and $S(K)$ the set of proper values of H and K respectively. We denote the set of proper vectors belonging to $\lambda \in S(H)$ or $\lambda \in S(K)$ by $E_\lambda(H)$ or $E_\lambda(K)$ respectively. We know that $S(K)$ is bounded, discrete and $E_\lambda(K)$ is finite-dimensional.

The purpose of this paper is to study in what cases $S(H)$ is bounded or discrete or, in the case of Hilbert spaces, $E_\lambda(H)$ contains finite number of orthogonal elements.

For this purpose, we see that the case when $H0 \neq 0$ is exceptional, because we have the following

Theorem 1. *Let $H=K\mathfrak{f}$ be defined on a Banach space and $H0\neq 0$. If there exist numbers $a>0$ and $b>0$ such that*

$$(\sharp) \qquad \| \mathfrak{f}\phi \| \leqq a+b\| \phi \|$$

for every $\phi \in R$, then $|\lambda| \geqq (a+b)\| K\|$ implies $\lambda \in S(H)$.

The proof is omitted, because this is an easy consequence of Schauder's fixed point theorem. In case of integral operators of Hammerstein type, defined on $L_p(p>1)$ or Orlicz spaces, the condition ($\sharp$) is equivalent to the fact $\mathfrak{f}$ is defined on the whole space. For this, we refer [1, 2].

In the sequel, we assume that $\mathfrak{f}0=0$.

§1. *Boundedness of* $S(H)$.

Theorem 2. *Let $H=K\mathfrak{f}$ be defined on a Banach space R. If the operator $\mathfrak{f}$ with ($\sharp$) be Fréchet-differentiable at* 0 *and the gradient $\nabla\mathfrak{f}0$ be continuous, then $S(H)$ is bounded.*

Proof. Since $\mathfrak{f}$ is Fréchet-differentiable, for any positive number $\varepsilon>0$ there exists $\delta>0$ such that

$$\| \mathfrak{f}\phi-(\nabla\mathfrak{f}0)\phi \| \leqq \varepsilon \| \phi \| \qquad \text{if } \| \phi \| < \delta.$$

Therefore, $\| \phi \| < \delta$ implies that

1) For $K\phi(s)=\int K(s,t)\phi(t)\,dt$ and $\mathfrak{f}\phi(t)=f(t,\phi(t))$, the integral operator $H\phi(s)=K\mathfrak{f}\phi(s)=\int K(s,t)\mathfrak{f}(t,\phi(t))\,dt$ is of Hammerstein type. In the remarks of this paper, we consider operators of this type.