104. On Ascoli Theorems

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Irving Glicksberg [2, Theorem 2] proved that every continuous real-valued function on a topological space X is bounded[1] if and only if every bounded equicontinuous family of functions in $C^*(X, R)$ has compact closure in the uniform topology on $C^*(X, R)$, the space of bounded continuous functions on X to the real numbers R. In this paper we obtain results related to this and other Ascoli type theorems.

Our terminology involving uniform spaces and topologies on functions spaces follows closely that of [3].

If X is a topological space, $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$, the set of continuous functions on X to Y, then $\{f_n\}$ is said to converge uniformly at x to f if, for $V \in \mathcal{V}$ there is a neighborhood U_x and an integer N such that $(f_n(y), f(y)) \in V$ whenever $n \geq N$ and $y \in U_x$. $\{f_n\}$ is said to be uniformly Cauchy at x if, for $V \in \mathcal{V}$ there is a neighborhood U_x and an integer N such that $(f_n(y), f_m(y)) \in V$ whenever $n, m \geq N$ and $y \in U_x$.

Lemma 1. If $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$ which is uniformly Cauchy at each point of X and converges pointwise to a function f, then f is continuous and $\{f_n\}$ converges uniformly at each point of X.

Proof. The pointwise uniform convergence follows as a special case of Theorem 10(b), page 229 [3]. The continuity of f follows easily.

Lemma 2. If X is pseudo-compact, $(Y, \mathcal{V})$ is a uniform space and $\{f_n\}$ is a sequence in $C(X, Y)$ which converges uniformly at each point of X to a function f, then $\{f_n\}$ converges uniformly on X to f.

Proof. If $\{f_n\}$ does not converge uniformly to f, then there is a sequence $\{x_j\}$ in X, a subsequence $\{f_{n_j}\}$, a positive number r and a pseudo-metric p in the gage of $\mathcal{V}$ such that $p(f_{n_j}(x_j), f(x_j)) > r$ for each j. We let

$$g_j(x) = \max(p(f_{n_j}(x), f(x)) - r, 0).$$

For each point x in X there is a neighborhood of x on which all except finitely many of the functions g_j vanish. Thus we can define the following continuous function

1) If every continuous real valued function on X is bounded, we say that X is pseudo-compact.