99. *Ergodic Theorems for Pseudo-resolvents*

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1. **The theorem.** Let X be a complete locally convex linear topological space, and $L(X, X)$ the algebra of all continuous linear operators on X into X. A pseudo-resolvent J_λ is a function on a subset $D(J)$ of the complex plane with values in $L(X, X)$ satisfying the resolvent equation

(1) $$J_\lambda - J_\mu = (\mu - \lambda) J_\lambda J_\mu.$$

We have, denoting by I the identity operator,

(2) $$(I - \lambda J_\lambda) = (I - (\lambda - \mu) J_\lambda)(I - \mu J_\mu)$$

and

(3) $$\lambda J_\lambda (I - \mu J_\mu) = (1 - \mu(\mu - \lambda)^{-1}) \lambda J_\lambda - \lambda(\lambda - \mu)^{-1} \mu J_\mu.$$

We see, by (1), that all J_λ, $\lambda \in D(J)$, have a common null space $N(J)$ and a common range $R(J)$. We also see, by (2), that all $(I - \lambda J_\lambda)$, $\lambda \in D(J)$, have a common null space $N(I-J)$ and a common range $R(I-J)$. $N(J)$ and $N(I-J)$ are closed linear subspace of X, but $R(J)$ and $R(I-J)$ need not be closed; we shall denote by $R(J)^a$ and $R(I-J)^a$ their closures respectively.

To formulate our ergodic theorems we prepare two lemmas.

Lemma 1. Let there exist a sequence $\{\lambda_n\}$ of numbers $\in D(J)$ such that

(4) $\lim_{n\to\infty} \lambda_n = 0$ and the family of operators $\{\lambda_n J_{\lambda_n}\}$ is equi-continuous.

Then we have

(5) $$R(I-J)^a = P(J) = \{x \in X; \lim_{n\to\infty} \lambda_n J_{\lambda_n} x = 0\},$$

and hence

(6) $$N(I-J) \cap R(I-J)^a = \{0\}.$$

Lemma 1′. Let there exist a sequence $\{\lambda_n\}$ of numbers $\in D(J)$ such that

(4)′ $\lim_{n\to\infty} |\lambda_n| = \infty$ and the family of operators $\{\lambda_n J_{\lambda_n}\}$ is equi-continuous.

Then we have

(5)′ $$R(J)^a = I(J) = \{x \in X; \lim_{n\to\infty} \lambda_n J_{\lambda_n} x = x\}$$

and hence

(6)′ $$N(J) \cap R(J)^a = \{0\}.$$

Our ergodic theorems read as follows.

Theorem 1. Let (4) be satisfied. Let, for a given $x \in X$, there exist a subsequence $\{\lambda_{n'}\}$ of $\{\lambda_n\}$ such that