132. On $L^{(k)}$-Transform and the Generalized Laplace Transform

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1.*) Let $f(\zeta)$ be the Laplace transform of the function $F(x)$:

$$f(\zeta)=L(F)=\int_0^\infty e^{-\zeta x}F(x)dx.$$

If this Laplace integral $L(F)$ is convergent for a complex number ζ, then this means that the function

$$\Phi(x)=\int_0^x e^{-\zeta x}F(x)dx=(e^{-\zeta x}F(x))*1$$

has a limit for $x\to\infty$. Here $g(x)*h(x)$ means

$$\int_0^x h(\tau)\,g(x-\tau)\,d\tau=\int_0^x h(x-\tau)\,g(\tau)\,d\tau.$$

If $L(F)$ is not convergent, then we consider the Cesàro's k th order (C, k) mean of Φ:

$$m_k(x)=\frac{k}{x^k}\{(e^{-\zeta x}F)*1*x^{k-1}\}=\frac{(e^{-\zeta x}F)*x^k}{x^k}$$

where k is a positive integer, and for $k=0$ we put $m_0(x)=\Phi(x)$.

If this mean has a limit for $x\to\infty$, then we say that the Laplace integral is (C, k) *convergent* for ζ, or for the values ζ *the* $L^{(k)}$*-transform of* F:

$$L^{(k)}(F)=\lim_{x\to\infty}\frac{(e^{-\zeta x}F)*x^k}{x^k}\quad\text{exists.}$$

The domain of convergence of $L^{(k)}(F)$ is a half plane: $\{\zeta\,|\,Re(\zeta)>\beta_k\}$ for some real number $\beta_k(-\infty\leq\beta_k\leq+\infty)$.

For any pair of positive integers k and k', such that $k'>k$, the inequality $\beta_{k'}\leqq\beta_k$ follows. So, for $k\to\infty$, β_k converges to the limit B. (B can be finite or $\pm\infty$.)

It can appear that $L^{(k)}(F)$ is convergent for the first time for some (large) k, whereas for the other (smaller) k', $L^{(k')}(F)$ do not converge.

The function $L^{(k)}(F)$ is analytic in the interior of the convergent domain $\{\zeta\,|\,Re(\zeta)>\beta_k\}$ and coincides with the functions $L^{(k')}(F)$ for $k'>k$, in the half plane defined by $\{\zeta\,|\,Re(\zeta)>\beta_k\}$. The totality of $L^{(k)}(F)$ for $k\geq 0$ defines a function $f(\zeta)$ in the half plane $\{\zeta\,|\,Re(\zeta)>B\}$ which we call L^∞-transform of F.

Using L^∞-transform, thus, we can examine the analytic continuation of $L(F)$ in the domain outside the axis of convergence of

*) See reference G. Doetsch [1]. In **1** our notations conform to those of G. Doetsch.