130. *Bordered Riemann Surface with Parabolic Double*

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(Comm. by K. KUNUGI, M.J.A., Nov. 13, 1961)

Let W be an open Riemann surface and $\{W_n\}_{n=0}^{\infty}$ be a sequence of regular subregions such that $\overline{W}_n \subset W_{n+1}$ and $W=\bigcup_{n=0}^{\infty} W_n$.[1] Let $u(p)$ be a harmonic function on $W-W_0$. For this u, construct the sequence $\{u_n\}_{n=1}^{\infty}$ of functions $u_n(p)$ continuous on $W-W_0$ and harmonic on $W_n-\overline{W}_0$ such that $u_n=u$ on ∂W_0 and $u_n=c$ on $W-W_n$, where c is a fixed constant. Assume

(1) $$\lim_n u_n(p)=u(p)$$

uniformly on each compact subset of $W-W_0$. For brevity, we denote this fact by $u=c$ *on the ideal boundary* ∂W *of* W. By using Dirichlet principle, it is easily seen that the Dirichlet integral $D_{W-\overline{W}_0}(u)$ is finite and

(2) $$\lim_n D_{W-\overline{W}_0}(u-u_n)=0.$$

It is also clear that

(3) $$|u_n(p)|,\ |u(p)|\leq \max(\max_{\partial W_0}|u(p)|,\ |c|)$$

on $W-W_0$. The Green function $g(p,q)$ with pole q in W_0 or the harmonic measure $w(p;W-W_0,\partial W)$ of ∂W is an important example of such a function u, i.e. $g(p,q)=0$ and $w(p;W-W_0,\partial W)=1$ on ∂W respectively. We put

$$m=\min_{\partial W_0} u(p) \text{ and } M=\max_{\partial W_0} u(p)$$

and assume that $c<m$ (or $c>M$). Choose an arbitrary number t such that

$$c<t<m \text{ (or } c>t>M)$$

and let R be a component of the open set $\{p\in W-\overline{W}_0;\ u(p)>t\}\cup\overline{W}_0$ (or $\{p\in W-\overline{W}_0;\ u(p)<t\}\cup\overline{W}_0$). It is easy to see that $R=W$ if and only if W is parabolic. Hence from now on we assume that W is hyperbolic. Then R is a bordered Riemann surface with border $\Gamma=\{p\in W;\ u(p)=t,\ du(p)\neq 0\}\cap\overline{R}$. Each component of the closure $\overline{\Gamma}$ of Γ is a piecewise analytic curve in W. Construct the double $\hat{R}$ of R along Γ. Z. Kuramochi pointed out the following fact:[2]

THEOREM. *The surface* $\hat{R}$ *is closed or parabolic.*

The proof of this theorem given by Kuramochi is based on his

1) For terminologies and notions not explained in this note, refer to Ahlfors-Sario's book, Riemann surfaces, Princeton, 1960.

2) Proc. Japan Acad., **32**, 25-30 (1955).