129. Re-topologization of Functional Space in Order that a Set of Operators will be Continuous

By Mitio NAGUMO

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The purpose of this note is to give a new topology to an abstract functional space in order that a set of linear operators, which need not be continuous primarily, will be continuous and that they will be extended to operators with the whole space as their domains. This may be regarded, in a sense, as an abstract generalization of the concept "distributions" by L. Schwartz [2].

This investigation has gotten the hint from the idea "negative norms" by P. D. Lax [1]. Here we shall give only general and abstract considerations. More concrete cases and applications will be given later on. A special case was considered by E. B. Cossi and the auther [3].

1. Let E be a *locally convex linear topological space*. We assume also that E is *bornologic*.

Let $\{T_\alpha\}$ $(\alpha\in\Omega)$ be a set of pre-closed linear operators from E into E, such that $D(T_\alpha)$, the domain of T_α, is dense in E. We assume also that the identical operator $1\in\{T_\alpha\}$.

Theorem 1. *Let T'_α be the adjoint operator of T_α and put $F=\bigcap_\alpha D(T'_\alpha)$. Assume that F is total on E, i.e., if $x\in E$ and $f(x)=0$ for all $f\in F$ then $x=0$. Then we can give a new topology to E, in such a way that all T_α will be continuous from E with the primary topology into E with the new topology.*

Proof: First let us give a new topology to F, in such a way that all T'_α will be continuous on $\tilde{F}$ into E', the dual of E, where $\tilde{F}$ means the same set as F but with the new topology.

For this purpose define the semi-norm $p'_{A,\alpha}$ on E' by

$$p'_{A,\alpha}(f)=\sup\{|T'_\alpha f(x)|;x\in A\} \qquad \text{for } f\in F,$$

where A is any bounded set in E. Then by the set of semi-norms $\{p'_{A,\alpha}\}$ $(A\in B(E),\ \alpha\in\Omega)$[1] F becomes a locally convex linear topological space $\tilde{F}$. As $1\in\{T'_\alpha\}$, the new topology of $\tilde{F}$ is stronger than the old topology of $F\subset E'$.

The operators T'_α are continuous on $\tilde{F}$ into E'. Because the topology of E' is given by the system of semi-norms $\{p'_A\}$ $(A\in B(E))$ defined by

$$p'_A(f)=\sup\{|f(x)|;x\in A\}$$

1) $B(E)$ denotes the system of all bounded sets in E.