124. Open Basis and Continuous Mappings

By Sitiro HANAI
Osaka University of Liberal Arts and Education

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Let X and Y be topological spaces and let $f(X)=Y$ be an open continuous mapping. B. Ponomarev [6] has recently obtained the following theorem: *if X has a point-countable open base and the inverse image $f^{-1}(y)$ is separable for each point y of Y, then Y has a point-countable open base.* It is interesting to know under what conditions the property of the open base of X will be preserved under the open (or closed) continuous mapping.

In this note, we shall deal with this problem.

1. Open basis and closed mappings. At the beginning of this section, we shall recall a definition. Let $\mathfrak{U}=\{U_\alpha\}$ be a open base of X. If $\mathfrak{U}$ is star-countable (or locally countable or point-countable), we say that $\mathfrak{U}$ is a star-countable (or locally countable or point-countable) open base.

Theorem 1. *If f is a closed continuous mapping from a topological space X with a star-countable open base onto a topological space Y such that the inverse image $f^{-1}(y)$ is connected and countably compact for each point y of Y, then Y has also a star-countable open base.*

Proof. Let $\mathfrak{U}=\{U_\alpha\}$ be a star-countable open base of X, then, by K. Morita's theorem [3] (or Yu. Smirnov's lemma [8]), we can see that X is decomposed in such a way that $X=\bigcup_{\gamma\in\Gamma}A_\gamma$, $A_\gamma\frown A_{\gamma'}=\phi$, $\gamma\neq\gamma'$, $\gamma,\gamma'\in\Gamma$, and $A_\gamma=\smile\{U_\alpha\in\mathfrak{U}_\gamma\}$ where $\mathfrak{U}_\gamma$ is a countable subfamily of $\mathfrak{U}$. Since $f^{-1}(y)$ is connected, we have $f^{-1}f(A_\gamma)=A_\gamma$ for each γ of Γ. Since f is a closed continuous mapping, $f(A_\gamma)$ is open. And moreover $f(A_\gamma)\frown f(A_{\gamma'})=\phi$ for $\gamma\neq\gamma'$. Since each A_γ is perfectly separable and $f^{-1}(y)$ is countably compact, $f^{-1}(y)$ is compact. Hence each $f(A_\gamma)$ is perfectly separable because f is closed and continuous. Therefore Y has a star-countable open base. This completes the proof.

Theorem 2. *Let f be a closed continuous mapping from a topological space X with a star-countable open base onto a topological space Y such that the point inverse image $f^{-1}(y)$ is compact (or separable and countably compact) for each point y of Y. Then Y has a star-countable open base if and only if any open covering of Y has a star-countable open refininement.*

Proof. As the "only if" part is obvious, we shall prove the "if" part. Let $\mathfrak{U}=\{U_\alpha\}$ be a star-countable open base of X. Since $f^{-1}(y)$