123. On the Spectra of Some Non-linear Operators. II

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In this note, we continue the study on the Hammerstein operators whose spectra contain no intervals. We denote the spectrum of a Hammerstein operator H by $S(H)$.[1)]

§1. Let $f_i(x)$ $(i=1,2,\cdots)$ be countable number of real-valued continuous functions with $f_i(0)=0$ defined on the whole real line, and k_i $(i=1,2,\cdots)$ be countable number of positive numbers. We define an operator H on l^2 of vectors $\phi=(x_1, x_2,\cdots)$ with $\sum_{i=1}^{\infty} x_i^2 < +\infty$ by

$$H\phi=(k_1f_1(x_1), k_2f_2(x_2),\cdots). \tag{1}$$

We assume that the range of H is also in l^2. This is of Hammerstein type, i.e. $H=K\mathfrak{f}$, where

$$\mathfrak{f}\phi=(f_1(x_1), f_2(x_2),\cdots)$$

and K is a matrix of diagonal form.

Theorem 1. *Let us assume that the functions* $g_i(x)=\dfrac{f_i(x)}{x}$ *be continuous. Then, for the operator* $H\phi$ *defined by* (1), *if* $S(H)$ *contains no intervals,* H *must be linear.*

Proof. When $k_1f_1(x_1)=\lambda x_1$ for some $x_1\neq 0$ and $\lambda\neq 0$, then we consider the vector $\phi_1=(x_1, 0, 0,\cdots)$ for which we have

$$\begin{aligned}H\phi_1&=(k_1f_1(x_1), k_2f_2(0), k_3f_3(0),\cdots)\\&=(\lambda x_1, 0, 0,\cdots)=\lambda\phi_1,\end{aligned}$$

namely, $\lambda\in S(H)$. Therefore, if the continuous function $g_1(x)$ takes two different values λ_1, λ_2 at points different from zero:

$$k_1g_1(x_1)=\lambda_1,\ k_1g_1(x_2)=\lambda_2;\quad x_1\neq 0,\ x_2\neq 0,$$

then, since $k_1g_1(x)$ takes every value between λ_1 and λ_2, $S(H)$ contains at least one interval. Namely, if $S(H)$ contains no intervals, $k_1g_1(x)$ must be constant, and hence it follows that

$$k_1f_1(x)=\lambda_1 x \qquad (-\infty<x<+\infty)$$

for a uniquely defined number λ_1. Similarly, we have

$$k_if_i(x)=\lambda_i(x) \qquad (-\infty<x<+\infty;\ i=2,3,\cdots).$$

Therefore, for $\phi=(x_1, x_2,\cdots)$ and $\psi=(y_1, y_2,\cdots)$, we have

$$\begin{aligned}H(x\phi+y\psi)&=(k_1f_1(xx_1+yy_1), k_2f_2(xx_2+yy_2),\cdots)\\&=(\lambda_1(xx_1+yy_1), \lambda_2(xx_2+yy_2),\cdots)\\&=x(k_1f_1(x_1), k_2f_2(x_2),\cdots)+y(k_1f_1(y_1), k_2f_2(y_2),\cdots)\end{aligned}$$

1) As was pointed out in the preceding paper [2], we need only to study the case when $H0=0$.