122. Further Measure-Theoretic Results in Curve Geometry

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1. **Extension of a previous result.** The final theorem of our recent note [4] was only given a sketched proof. We shall now prove it completely, extending it at the same time to the following form which is slightly more general.

THEOREM. *If I is an interval (of any type) on which a curve φ, situated in $\boldsymbol{R}^m$, is both continuous and rectifiable, then $\Xi(\varphi; E)=\Gamma(\varphi; E)$ for any subset E of I whatsoever.*

PROOF. We may clearly suppose I an endless interval, so that φ is continuous at all points of I. Let us denote by $\mathfrak{N}$ the class of all the Borel sets $X\subset I$ fulfilling the relation $\Xi(\varphi; X)=\Gamma(\varphi; X)$, and by $\mathfrak{M}$ the class of all convex subsets of I. Each set $(\mathfrak{M})$ being then either void, or a one-point set, or an interval, we see as at the end of [4] that the class $\mathfrak{N}$ contains $\mathfrak{M}$. As may be readily verified further, $\mathfrak{M}$ is a primitive class in I (see p. 116 of our paper [1] for the terminology). In other words, $\mathfrak{M}$ satisfies the following three conditions: (i) the interval I belongs to $\mathfrak{M}$; (ii) if $A\in\mathfrak{M}$ and $B\in\mathfrak{M}$, then $AB\in\mathfrak{M}$; (iii) if $A\in\mathfrak{M}$, there is a disjoint infinite sequence Δ of sets $(\mathfrak{M})$ such that $I-A=[\Delta]$. Consequently, in conformity with Theorem 1 of [1], the smallest additive class (in I) containing the class $\mathfrak{M}$ coincides with the smallest normal class containing $\mathfrak{M}$ (see Saks [7], p. 83, for the terminology). But, taking into account the rectifiability of φ on I, we find easily that $\mathfrak{N}$ is a normal class. It follows at once that $\mathfrak{N}$ coincides with the Borel class in I, so that our assertion holds at least whenever E is a Borel subset of I.

Let us turn now to the case of general E. As it will follow from the lemma to be soon established below, we can enclose E in a Borel set $E_0\subset I$ such that $\Gamma(\varphi; E)=\Gamma(\varphi; E_0)$. Since $\Gamma(\varphi; E_0)=\Xi(\varphi; E_0)$ by what has already been proved, we obtain $\Gamma(\varphi; E)\geqq\Xi(\varphi; E)$. This, combined with the lemma of [4]§2, gives finally $\Gamma(\varphi; E)=\Xi(\varphi; E)$.

LEMMA. *If a curve φ is continuous at all points of a set E, we can enclose E in a set H of the class $\mathfrak{G}_\delta$ such that $\Gamma(\varphi; H)=\Gamma(\varphi; E)$.*

PROOF. We may plainly assume $\Gamma(\varphi; E)$ finite. To simplify our notations, let us write $\Phi(X)=\mathrm{d}(\varphi[X])$ for each set X. Given any natural number n, the set E has an expression as the join of an infinite sequence $\Delta_n=\langle X_1^{(n)}, X_2^{(n)},\cdots\rangle$ of its subsets such that $\mathrm{d}(X_i^{(n)})<\varepsilon$ for $i=1, 2,\cdots$ and $\Phi(\Delta_n)<\Gamma(\varphi; E)+\varepsilon$, where and below we write $\varepsilon=n^{-1}$