143. Functional-Representations of Normal Operators in Hilbert Spaces and their Applications

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In this paper we have mainly two aims: one is to express a normal operator in a Hilbert space by continuous linear functionals associated with all elements of a complete orthonormal set in that space and the other is to construct a normal operator with the arbitrarily prescribed point spectrum. We can yet treat these two problems at the same time.

Definition. Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let $\{\varphi_\nu\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\cdots}$ both be incomplete orthonormal infinite sets which have no element in common and together form a complete orthonormal set in $\mathfrak{H}$; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrarily prescribed bounded sequence in the complex plane; let (u_{ij}) be an infinite unitary matrix with $|u_{jj}|\neq 1, j=1,2,3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$; let N be the operator defined by

$$Nx=\sum_{j=1}^{\infty}\lambda_\nu(x,\varphi_\nu)\varphi_\nu+c\sum_{\mu=1}^{\infty}(x,\psi_\mu)\Psi_\mu$$

for every $x\in\mathfrak{H}$ and an arbitrarily given constant c; let L_f be the continuous linear functional associated with an arbitrary element f in $\mathfrak{H}$; and let the operator N and the element Nx, defined above, be denoted symbolically by

$$(1)\qquad N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}$$

and

$$(2)\qquad Nx=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}(x)+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}(x)$$

respectively. Then the sum of the two series in the right-hand side of (1) is called "the functional-representation of the operator N".

Theorem 1. The functional-representation of the operator N defined by (1) converges uniformly and N is a bounded normal operator with the point spectrum $\{\lambda_\nu\}$ on $\mathfrak{H}$. In addition, putting $M=\max(S,|c|^2)$ where $S=\sup_\nu|\lambda_\nu|^2$, $||N||=\sqrt{M}$.

Proof. Since, by hypotheses, a complete orthonormal set is formed by the two sets $\{\varphi_\nu\}$ and $\{\psi_\mu\}$, we have for every $x\in\mathfrak{H}$

$$x=\sum_{\nu=1}^{\infty}a_\nu\varphi_\nu+\sum_{\mu=1}^{\infty}b_\mu\psi_\mu,$$