6. On the Functional-Representations of Normal Operators in Hilbert Spaces

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Let $\mathfrak{H}$ be the complex abstract Hilbert space which is complete, separable, and infinite dimensional; let $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ both be incomplete orthonormal sets in $\mathfrak{H}$ which have no element in common and together form a complete orthonormal set in that space; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be an arbitrarily prescribed bounded sequence in the complex plane; let $\{u_{ij}\}$ be an infinite unitary matrix with $|u_{jj}|\neq 1$, $j=1,2,3,\cdots$; let $\Psi_\mu=\sum_{j=1}^{\infty}u_{\mu j}\psi_j$; let L_x be the continuous linear functional associated with an arbitrary $x\in\mathfrak{H}$; and let $y\otimes L_x$ be the operator defined by $(y\otimes L_x)z=(z,x)y$ for an arbitrarily given $y\in\mathfrak{H}$ and for every $z\in\mathfrak{H}$. Then, with respect to the operator N defined as

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu},$$

where c is an arbitrarily given complex constant, I have proved in Vol. 37, No. 10 (1961) of Proceedings of the Japan Academy that not only the right-hand side converges uniformly, but that also N is a bounded normal operator with point spectrum $\{\lambda_\nu\}$ in $\mathfrak{H}$, and have defined the expression of the right-hand side as "the functional-representation of N".

The purpose of this paper is to prove that conversely every bounded normal operator N in $\mathfrak{H}$ is essentially expressible by such an infinite series of the continuous linear functionals associated with all the elements of a complete orthonormal set in $\mathfrak{H}$ as described above.

Theorem A. Let N be a bounded normal operator in $\mathfrak{H}$; let $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ be its point spectrum (inclusive of the multiplicity of each eigenvalue of N); let $\{\varphi_\nu\}_{\nu=1,2,3,\dots}$ be an orthonormal set determining the subspace $\mathfrak{M}$ determined by all the eigenelements of N, such that φ_ν is a normalized eigenelement corresponding to an arbitrary eigenvalue λ_ν of N; let $\{\psi_\mu\}_{\mu=1,2,3,\dots}$ be an orthonormal set determining the orthogonal complement $\mathfrak{N}$ of $\mathfrak{M}$; and let L_f be the continuous linear functional associated with any $f\in\mathfrak{H}$. Then $\|N\psi_\mu\|^2$, $\mu=1,2,3,\cdots$, assume the same value, which will be denoted by σ; and if we choose arbitrarily a complex constant c with absolute value $\sqrt{\sigma}$ and put $\Psi_\mu=\sum_j u_{\mu j}\psi_j$, where $u_{\mu j}=(N\psi_\mu,\psi_j)/c$ and $\sum_j$