1. On the Measure-Bend of Parametric Curves

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1. Curves straightenable on a set. In the present continuation of our recent note [5] we shall derive some further measure-theoretic properties of parametric curves. Throughout the note the space $\boldsymbol{R}^m$ will be assumed at least 2-dimensional, while all the curves considered will be defined over $\boldsymbol{R}$ (unless stated to the contrary) and situated in $\boldsymbol{R}^m$. A curve $\varphi(t)$ will be termed *straightenable* (or *of bounded bend*) on a set E of real numbers iff the bend $\Omega(\varphi; E)$ is finite, and *locally straightenable* (or *of locally bounded bend*) iff φ is straightenable on all linear closed intervals. Let us begin our argument with a lemma which extends [1]§64.

LEMMA. *If a curve φ is straightenable on a set E as well as bounded on E, it is rectifiable on the same set. In consequence, a locally straightenable curve is locally rectifiable whenever it is locally bounded.*

REMARK. Simple examples show that the boundedness of φ on E is essential for the validity of the assertion (cf. the remark of [1]§64).

PROOF. By change of parameter if necessary, we may suppose without loss of generality that E is a bounded set. Let I_0 denote generically an open interval. We shall show in the first place that if $\Omega(\varphi; I_0E)<\pi/3$, the curve φ is rectifiable on I_0E and we have $L(\varphi; I_0E)\leqq 2\mathrm{d}(\varphi[I_0E])$, where for any set X in $\boldsymbol{R}^m$ we denote by $\mathrm{d}(X)$ the diameter of X. For this purpose we may suppose $L(I_0E)$ positive. It suffices to derive $L(IE)\leqq 2|\varphi(I)|$ for each closed interval I contained in I_0 and whose endpoints belong to E. For it is obvious, by definition of length, that $L(I_0E)$ is the supremum of $L(IE)$. We now distinguish two cases according as the increment $\varphi(I)$ vanishes or not. If $\varphi(I)=0$, then φ must be constant on the set IE and hence $L(IE)=0=2|\varphi(I)|$; indeed we should otherwise get the evident contradiction $\Omega(I_0E)\geqq\Omega(IE)\geqq\pi$. If on the other hand $\varphi(I)\neq 0$, then $L(IE)\leqq 2|\varphi(I)|$ follows easily by an argument similar to that of [1]§63. We leave the details to the reader.

Writing $\theta=\Omega(I_0E)$ for an arbitrary $I_0=(a, b)$, we shall further show that there exists in I_0 a point c such that $\Omega((a, c)\cdot E)\leqq\theta/2$ and $\Omega((c, b)\cdot E)\leqq\theta/2$. Of course we need only consider the case $\theta>0$. It is clear that (i) the supremum of the bend $\Omega(JE)$, where J ranges