11. *On the Weak Definability in Set Theory*

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We refer to [4] and [5] as to the notions and notations throughout this paper. Here we restate some of them. A "set theory" means a set theory in the first order predicate calculus, containing only the predicate $\in$, logical symbols, bound variables and finitely or infinitely many individual constants. If T is a set theory in this sense containing $a_0, a_1, \cdots$ as individual constants, we call T a set theory with $a_0, a_1, \cdots$. Let T be a set theory with $a_0, a_1, \cdots$ and B_T be the class consisting of all $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$, where $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$ contains only logical symbols, the predicate $\in$, bound variables and $a_0, a_1, \cdots$. $\{x\}\mathfrak{A}(x, a_0, a_1, \cdots)$ will be abbreviated as $\{x\}\mathfrak{A}(x)$ if no confusion is to be feared. T is called 'definite', if it satisfies the following conditions: 1) T is complete. (I.e. for any closed formula $\mathfrak{A}$ (which, more precisely, should be written as $\mathfrak{A}(a_0, a_1, \cdots)$) either $\mathfrak{A}$ or $\neg\mathfrak{A}$ belongs to T.) 2) If $\exists x\mathfrak{A}(x)$ belongs to T, then there exists a formula $\exists x\mathfrak{B}(x)$ such that $\forall x\forall y(\mathfrak{B}(x)\wedge\mathfrak{B}(y) \vdash x=y)$ and $\exists x(\mathfrak{A}(x)\wedge\mathfrak{B}(x))$ belong to T.

Let $\{x\}\mathfrak{A}(x)$ and $\{x\}\mathfrak{B}(x)$ belong to B_T. We say '$\{x\}\mathfrak{B}(x)$ belongs to the same class with $\{x\}\mathfrak{A}(x)$ relative to T', if and only if $\forall x(\mathfrak{A}(x) \dashv\vdash \mathfrak{B}(x))$ belongs to T. The class which contains $\{x\}\mathfrak{A}(x)$ is written $(\{x\}\mathfrak{A}(x))$ and $\{x\}\mathfrak{A}(x)$ is said to represent the class. A class $(\{x\}\mathfrak{A}(x))$ is said to be definite with respect to T, if $\exists x\mathfrak{A}(x)$ and $\forall x\forall y(\mathfrak{A}(x)\wedge\mathfrak{A}(y) \vdash x=y)$ belong to T. $A(T)$ is defined to be the set of all the definite classes. Let $(\{x\}\mathfrak{A}(x))$ and $(\{x\}\mathfrak{B}(x))$ be two elements of $A(T)$. Then

$$(\{x\}\mathfrak{A}(x)) \in^*_T (\{x\}\mathfrak{B}(x))$$

is defined to mean that '$\exists x\exists y(\mathfrak{A}(x)\wedge\mathfrak{B}(y)\wedge x\in y)$ belongs to T'.

In [5] we considered a set theory $T_C(a)$. It contains all the elements of a set a in C (='Cantor's Absolute') as individual constants, and consists of all the formulas which are true in C. Here we consider the set theory $T_C(On)$ which contains all ordinal numbers as individual constants.

A set in C is called *weakly definable* (or a weakly definable set) if it is definable in $T_C(On)$.

We present the following two statements as the first and the second weak definability principles:

1. Every set is weakly definable.
2. Every non-empty weakly definable set contains a weakly