10. On the Reduced Measure-Bend of Curves

By Kanesiroo ISEKI

Department of Mathematics, Ochanomizu University, Tokyo

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1. A property of B-measurable curves. Continuing our recent note [4], we shall commence the present study with the following result, whose proof is based on an argument essentially the same as that of the proof for Theorem (4.3) stated on p. 113 of Saks [5] and concerning the measurability of Dini derivates. The underlying space $\boldsymbol{R}^m$ will be assumed at least 2-dimensional unless specified otherwise.

THEOREM. *Given a B-measurable curve φ and a B-measurable spheric curve γ (both situated in $\boldsymbol{R}^m$), let K be the set of the points t at which $\gamma(t)$ is a right-hand derived direction for φ. Then K is a Borel set.*

REMARK. Needless to say, a curve is called B-measurable iff all its coordinate functions are B-measurable.

PROOF. For each pair p, q of natural numbers let us define a point set $K(p,q)\subset\boldsymbol{R}$ as follows: a point t of $\boldsymbol{R}$ belongs to $K(p,q)$ when and only when there exists an x such that

$$t+q^{-1}<x<t+p^{-1},\quad |\varphi(x)-\varphi(t)|>q^{-1},\quad \gamma(t)\diamond[\varphi(x)-\varphi(t)]<p^{-1}.$$

We see at once that $K(p,q)$ is descending and ascending in p and q respectively and that the set K of the assertion may be written

$$K=\lim_p \lim_q K(2p,q)=\lim_p \lim_q K(p,2q).$$

Since φ is B-measurable, we can easily associate with each $n=1,2,\cdots$ a B-measurable curve $\varphi_n(t)$ such that the image $\varphi_n[\boldsymbol{R}]$ is countable and that $|\varphi_n(t)-\varphi(t)|<n^{-1}$ for every $t\in\boldsymbol{R}$. Similarly there is a B-measurable spheric curve $\gamma_n(t)$ such that $\gamma_n[\boldsymbol{R}]$ is countable and $|\gamma_n(t)-\gamma(t)|<n^{-1}$ for every t. So that the two sequences of curves, $\langle\varphi_1(t),\varphi_2(t),\cdots\rangle$ and $\langle\gamma_1(t),\gamma_2(t),\cdots\rangle$, tend respectively to $\varphi(t)$ and $\gamma(t)$ uniformly on the real line. Let us now replace, in the above definition of the set $K(p,q)$, the curves φ and γ by φ_n and γ_n respectively and write $K_n(p,q)$ for the resulting set. We then denote for short the two limits

$$\operatorname*{lim\text{-}inf}_n K_n(p,q) \quad\text{and}\quad \operatorname*{lim\text{-}sup}_n K_n(p,q)$$

by $U(p,q)$ and $V(p,q)$ respectively and find readily that

$$K(2p,q)\subset U(2p,q)\subset V(2p,q)\subset K(p,2q).$$

From this we deduce, by what has already been proved, that

$$K\subset\lim_p \lim_q U(2p,q)\subset\lim_p \lim_q V(2p,q)\subset K.$$