24. Further Properties of Reduced Measure-Bend

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1. Completion of a previous result. We shall be concerned with curves defined on the real line $\boldsymbol{R}$ and situated in $\boldsymbol{R}^m$, where we assume $m \geqq 2$ unless stated otherwise. By sets, by themselves, we shall understand subsets of $\boldsymbol{R}$. Continuing our recent note [6], let us begin with a theorem which completes part (ii) of the theorem of [5]§3.

THEOREM. *Given a curve φ and a set E, suppose that $\Omega_*(\varphi; M)$ vanishes for every countable set $M \subset E$. Then*

$$\Upsilon(\varphi; E) = \Omega_*(\varphi; E) \leqq \Omega_*(\psi; E)$$

for each curve ψ which coincides on E with φ.

PROOF. The lemma and the theorem of [6]§2 require respectively that $\Upsilon(\psi; E) \leqq \Omega_*(\psi; E)$ and $\Upsilon(\varphi; E) = \Omega_*(\varphi; E)$. But our hypothesis on the curve ψ clearly implies $\Upsilon(\varphi; E) = \Upsilon(\psi; E)$. Hence the result.

REMARK. The above theorem has a counterpart in length theory, as follows. (The proof is not difficult and may be left to the reader.)

Given a curve φ and a set E, suppose that $L_*(\varphi; M) = 0$ holds for every countable set $M \subset E$. Then $\Xi(\varphi; E) = L_*(\varphi; E) \leqq L_*(\psi; E)$ for each curve ψ which coincides on E with φ.

Here the space in which the two curves lie may exceptionally be of any dimension.

2. Another definition of reduced measure-bend. By the *essential measure-bend* of a curve φ over a set E, we shall mean the infimum of the measure-bend $\Omega_*(\psi; E)$, where ψ is any curve which coincides on E with φ. The notation $\Omega_0(\varphi; E)$ will be used for it. In terms of this quantity we shall now give a second definition to the notion of reduced measure-bend. Indeed the theorem of [4]§2 has the following analogue.

THEOREM. *Given a curve φ and a set E, represent E in any manner as the join of a sequence Δ of subsets and write $\Upsilon_0(\varphi; E)$ for the infimum of the sum $\Omega_0(\varphi; \Delta)$. Then $\Upsilon_0(\varphi; E) = \Upsilon(\varphi; E)$.*

PROOF. On account of the lemma of [6]§2 we have in the first place $\Upsilon(\varphi; E) = \Upsilon(\psi; E) \leqq \Omega_*(\psi; E)$ for every curve ψ considered above. It ensues that $\Upsilon(\varphi; E) \leqq \Omega_0(\varphi; E)$, where we observe that E may be replaced by any other set. Therefore $\Upsilon(\varphi; E) \leqq \Upsilon(\varphi; \Delta) \leqq \Omega_0(\varphi; \Delta)$ for every Δ, and from this we infer that $\Upsilon(\varphi; E) \leqq \Upsilon_0(\varphi; E)$. The deduc-