23. An Asymptotic Property of a Gap Sequence

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1. **Introduction.** Let $f(t)$ be a real measurable function satisfying

(1.1) $$f(t+1)=f(t),\quad \int_0^1 f(t)dt=0 \text{ and } \int_0^1 f^2(t)dt<+\infty,$$

and $\{n_k\}$ be a lacunary sequence of positive integers, that is,

(1.2) $$n_{k+1}/n_k>q>1.$$

Then the sequence of functions $\{f(n_k t)\}$, although themselves not independent, exibits the properties of independent random variables (c.f. [3]). In [2] Professor S. Izumi proved that if $f(t)$ satisfies certain smoothness conditions, then $\{f(2^k t)\}$ obeys the law of the iterated logarithm. However if we put $f(t)=\cos 2\pi t+\cos 4\pi t$ and $n_k=2^k-1$, then, by the theorem of Erdös and Gál [1], we have,

$$\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{N\log\log N}}\sum_{k=1}^N f(n_k t)=2\cos\pi t,\quad \text{a.e. in } t.$$

This shows that $\{f(n_k t)\}$ does not necessarily obey the law of the iterated logarithm even if $f(t)$ is a trigonometric polynomal.

In §§ 2–4 we shall prove the following

Theorem. Let $f(t)$ and $\{n_k\}$ satisfy (1.1) *and* (1.2) *respectively and $f(t)$ be a function of* Lip $\alpha, 0<\alpha\leq 1$. *Then we have,*

$$\overline{\lim_{N\to\infty}}\frac{1}{\sqrt{N\log\log N}}\sum_{k=1}^N f(n_k t)\leq C,\quad \text{a.e. in } t,$$

where C is a positive constant depending on $f(t)$ and q in (1.1).

2. **Preliminary.** From now on let $f(t)$ and $\{n_k\}$ satisfy the conditions of the theorem. For simplicity of writing we may assume that

$$f(t)\sim\sum_{k=1}^\infty c_k\cos 2\pi kt.$$

The proof is the same in the general cases as we can see by writing

$$a_k\cos 2\pi kt+b_k\sin 2\pi kt=\rho_k\cos 2\pi k(t-\xi_k).$$

In this paragraph let N be any fixed integer satisfying

(2.1) $$q^N>3N^\beta$$

where β is a positive constant such that $\alpha\beta=6$.

Let us put, for $m=0,1,\cdots,$

(2.2) $$g(t)=\sum_{k=1}^{N^\beta}c_k\cos 2\pi kt \text{ and } U_m(t)=\sum_{l=Nm+1}^{N(m+1)}g(n_l t).$$

Since $f(t)\in\text{Lip}\,\alpha$ and $\alpha\beta=6$, we have for some constant A,

(2.3) $$|f(t)-g(t)|<AN^{-\alpha\beta}\log N\leq AN^{-6}\log N,\quad \text{for all } t,$$