39. *On a Variant of Hausdorff Measure-Bend*

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1. Conditions for countable straightenableness and countable rectifiability. The present article is a continuation of our recent notes which have appeared in these Proceedings. The underlying space $\boldsymbol{R}^m$ will be assumed throughout to be at least two-dimensional.

We begin by stating the following result which is analogous to Theorem (9.1) on p. 233 of Saks [6] and which may be established as for that theorem with the aid of the category theorem of Baire.

THEOREM. *In order that a curve which is continuous on a nonvoid closed set E of real numbers, be countably straightenable [or countably rectifiable] on E* (see [5]§4 and [1]§2), *it is necessary and sufficient that every nonvoid closed subset of E should contain a portion on which the curve is straightenable [rectifiable].*

There is another condition sufficient for countable rectifiability which is closely related to Theorem (10.8) of Denjoy given on p. 237 of Saks [6]. For this purpose we have to introduce a few definitions. A curve φ situated in $\boldsymbol{R}^m$ will be said to be *conic on the right [on the left]* at a point $t_0 \in \boldsymbol{R}$, iff (i.e. if and only if) it is possible to choose a number δ of the interval $0<\delta<\pi/2$ and a nonvanishing vector p of the space $\boldsymbol{R}^m$, in such a manner that whenever a closed interval I of length $|I|<\delta$ has t_0 for its left-hand [right-hand] extremity and moreover the increment $\varphi(I)$ of the curve φ over I does not vanish, the angle between $\varphi(I)$ and p is less than $(\pi/2)-\delta$. We shall further term φ to be *unilaterally conic* at t_0 iff it is conic on the side (right or left) at t_0.

Our condition may now be set forth in the following form.

THEOREM. *If at every point t of a linear set E, except perhaps at the points of a countable subset, a curve φ is unilaterally conic, then φ is countably rectifiable on E.*

PROOF. Let A be the set of the points of $\boldsymbol{R}$ at which the curve φ is conic on the right. It is certainly enough to show that φ is countably rectifiable on A. Consider the rational vectors (i.e. having rational components) of $\boldsymbol{R}^m$ other than the zero vector. Noting that they are countable in number, we arrange all of them in a distinct infinite sequence $p_1, p_2, \cdots$. For each natural number n we denote by A_n the set of the points $t \in \boldsymbol{R}$ such that $|\varphi(t)|<n$ and further that, for every closed interval I whose length is $<1/n$ and whose left-