§ 1. Euler and circle methods of summability of Fourier series. Here the author wishes to discuss the circle method of summability and other quasi-Hausdorff methods of summability of Fourier series. At the beginning we remember the Euler method of summability. It associates with a given sequence \( \{s_n\} \) the means

\[
\sigma_{n,r} = \sigma_n = \frac{1}{n} \sum_{i=0}^{n-1} r^i \cdot (1-r)^n \cdot s_i, \quad n=0,1,2, \ldots ,
\]

where \( r \) is a constant which satisfies \( 0 < r \leq 1 \). We denote this method as \((\varepsilon, r)\). The case \( r = 1 \) corresponds to ordinary convergence. The Lebesgue constants for this method of Fourier series are given by Lebesgue [1] and A. E. Livingston [2].

Theorem 1. The Lebesgue constants for the \((\varepsilon, r)\) method are given by

\[
L(n; r) = \frac{2}{\pi^2} \log \frac{2n r}{1-r} + A + o(1), \quad n \to \infty, \quad \text{where}
\]

\[
A = -2C + \frac{2}{\pi} \int_0^1 \sin u \, du - \frac{2}{\pi} \int_1^\infty \left( \frac{2}{\pi} - \left| \sin \frac{u}{\pi} \right| \right) \frac{du}{u}.
\]

\( C \) is the Euler-Mascheroni constant.

The Gibbs phenomenon of the Fourier series \( \sum_{n=1}^\infty \frac{\sin nt}{n} \) for this method are investigated by O. Szász [3].

Theorem 2. If we put \( s_0 = 0, \quad s_n = \sum_{i=1}^n \frac{\sin vt}{\nu} \), then we have

\[
\lim_{nt \to \tau} \sigma_n(t_n) = \int_0^\tau \frac{\sin y}{y} \, dy, \quad \text{as} \quad nt_n \to \tau \quad \text{and} \quad nt_n^2 \to 0.
\]

On the other hand the circle method of summability associates with a given sequence \( \{s_n\} \) the means

\[
\sigma_{n,r}^* = \frac{1}{n} \sum_{i=0}^{n-1} r^i \cdot (1-r)^n \cdot s_i, \quad n=0,1,2, \ldots ,
\]

where \( r \) is a constant which satisfies \( 0 < r \leq 1 \). The case \( r = 1 \) corresponds to ordinary convergence. We denote this method as \((\gamma, r)\). The Lebesgue constants for this method of Fourier series are given by the author [4].

Theorem 3. The Lebesgue constants for the \((\gamma, r)\) method are given by