53. *A Remarkable Divergent Fourier Series*

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It is very well known that A. N. Kolmogorov [2] was the first to construct an example of a function $f(x) \in L(0, 2\pi)$ whose Fourier-Lebesgue trigonometric series diverges almost everywhere. Later he constructed a Fourier-Lebesgue series which diverges unboundedly everywhere [3]. But the Fourier series given by Kolmogorov is not a Fourier series of a function $f(x) \in L \log^+ L$, since its conjugate series is not a Fourier series.[1] The next step forward was made by G. H. Hardy and W. W. Rogosinski [1]. They constructed an almost everywhere divergent Fourier series whose conjugate series is also a Fourier series.[2]

In another direction, K. Zeller [8] gave a method to construct a Fourier series which converges on an arbitrary set $E \subset (0, 2\pi)$ of the type F_σ (denumerable sum of closed sets) and diverges unboundedly on $E_1 = [0, 2\pi) - E$. Recently L. V. Taikov [6] constructed a Fourier series which converges on $E \subset [0, 2\pi)$ of the type F_σ and diverges unboundedly everywhere on $E_1 = [0, 2\pi) - E$ such that the conjugate series is also a Fourier series.

It is natural to inquire whether the Fourier series of a function $f(x)$ belonging to $L^2(0, 2\pi)$ converges almost everywhere. This was conjectured by N. N. Lusin in the positive sense some forty-five years ago,[3] but it has neither been proved nor been disproved. To attack this difficult problem, it is of interest to observe the maximum speed at which a Fourier series may diverge unboundedly almost everywhere. If there exists a Fourier series which diverges very fast, we might think that the Lusin's conjecture could not be true. Concerning to this point, A. Zygmund ([10], p. 308) conjectured that for any sequence of positive numbers $\lambda_n = o(\log n)$, $n \to \infty$, there is an $f \in L$ such that at almost every point x we have $S_n(x; f) > \lambda_n$ for infinitely many n, where $S_n(x; f)$ denotes the nth partial sum of the Fourier

1) See, for example, [10] p. 308 and [7] Theorem 9. But, the series considered in [7] §3 is different from the original series defined by Kolmogorov, since the function $\phi_n(x)$ defined in [7] §3 is not a Féjer kernel. Each function $f(x)$ of the class denoted by $L \log^+ L$ is such that $|f(x)| \log^+ |f(x)| \in L(0, 2\pi)$.

2) In the English translation of [6]: Soviet Math., **6**, No. 2, p. 347, it is stated that Hardy and Rogosinski constructed an everywhere divergent Fourier series whose conjugate series is also a Fourier series, but it has been wrongly translated, cf. also [5].

3) See [4] p. 219.