69. Projective Limits and Metric Spaces with u-Extension Properties

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(Comm. by K. KUNUGI, M.J.A., July 12, 1962)

A metric space is said to *have a u-extension property* if any uniformly continuous real map defined on any subspace can always be extended uniformly over the whole space. Corson and Isbell [6] proved the theorem that a metric space has a u-extension property if and only if its completion is a projective limit [5] of fine metric spaces. We know [1, 3] some conditions characterizing a metric space with a u-extension property. Using the conditions and applying the idea of Flachsmeyer [7], we are, in this note, going to prove the same theorem with a somewhat simpler projective system.

We know (Theorem 2, [1]) that *a metric complete space S has a u-extension property if and only if, for any natural number n, there is a compact subset K_n such that for any open set G containing K_n there is a natural number m satisfying $V_{1/m}^{\infty}(x) \subset V_{1/n}(x)$ for every point $x \notin G$*, where $V_{1/n}$ is the entourage $\{(x, y); d(x, y) < 1/n\}$ of the uniform structure of the space and $V_{1/m}^{\infty}(x)$ is the set of all points which are joined with x by $V_{1/m}$-chains.

K_n in this statement is taken as the set of all points x satisfying $V_{1/i}^{\infty}(x) \not\subset V_{1/n}(x)$ for any i [3]. For each $x \notin K_n$, we take the least natural number $i(n, x)$ of numbers j with $V_{1/j}^{\infty}(x) \subset V_{1/n}(x)$, and put

$$H_n(x) = V_{1/i(n,x)}^{\infty}(x).$$

(1) *$H_m(y) \supset H_n(x)$ if and only if $H_m(y) \cap H_n(x) \neq \phi$ and $i(m, y) \leq i(n, x)$.*

In fact, if $H_m(y) \supset H_n(x)$ and $i(m, y) > i(n, x)$, then $H_n(x) \supset V_{1/i(n,x)}^{\infty}(y)$, and so $V_{1/i(n,x)}^{\infty}(y) = V_{1/i(m,y)}^{\infty}(y)$, which contradicts the definition of $i(m, y)$.

Hence there is the greatest $H_n(y)$ containing $H_n(x)$ whose $i(n, y)$ is the least of $i(n, z)$ with $H_n(z) \supset H_n(x)$, such the $H_n(y)$ is denoted by $G_n(x)$.

(2) *$G_n(x) \neq G_n(y)$ implies $G_n(x) \cap G_n(y) = \phi$.*

We put

$$J_n = K_n - \bigcup_{x \notin K_n} G_n(x)$$

and have the equivalent relation R_n on S defined by the cover

$$\alpha_n = \{(p), G_n(x); p \in J_n, x \in S - K_n\},$$

where (p) is the singleton, namely, $xR_n\, y$ if no member of α_n includes