65. On Bertrand's Problem in an Arithmetic Progression

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In this note, we shall prove the following

Theorem. There exists a positive constant c such that, if
$$x \geqq \exp(c \log k \log\log k)$$
and k is sufficiently large, then
$$\pi(2x; k, l) - \pi(x; k, l) > 0$$
is true for all l, satisfying $(k, l)=1$.

We shall use the same notations and symbols as in Prachar's book [Primzahlverteilung, Springer, 1957].

If $x \geqq \exp(k)$, the theorem is true by Theorem 8.3 in p. 144 or Theorem 3.2 in p. 323 of the book. Hence, we assume that

(1) $$\exp(c \log k \log\log k) \leqq x \leqq \exp(k).$$

Consequently,

(2) $$c \log k \log\log k \leqq \log x, \quad \frac{c}{2} \log\log x \leqq \frac{\log x}{\log k},$$

if k is sufficiently large.

We know from the results of Page [see IV, §5 and §6] and Linnik [see X, §3] that there exists a positive constant c_0 such that there are no zeros of any L-function mod k in the rectangle
$$1 - \frac{c_0}{\log k} \leqq \sigma \leqq 1, \quad |t| \leqq k^4$$
except possible one real zero β_1 of a particular L-function formed with a real character. Further if we put
$$\delta_0 = \begin{cases} 1-\beta_1 & \text{if the exceptional zero exists,} \\ \dfrac{c_0}{\log k} & \text{otherwise,} \end{cases}$$
then the rectangle
$$1 - \lambda(k) \leqq \sigma \leqq 1, \quad |t| \leqq k^4$$
contains no zero of any L-function mod k except β_1, where

(3) $$\lambda(k) = \frac{c_0}{\log k} \log \frac{c_0 e}{\delta_0 \log k}.$$

Now the constant c in the theorem will be given such that

(4) $$cc_0 \geqq 20.$$

Proof. From p. 321 of Prachar's book, we obtain
$$\varphi(k)\{\psi(2x; k, l) - \psi(x; k, l)\}$$