97. On a Product of Summability Methods

By Kazuo ISHIGURO

Department of Mathematics, Hokkaido University, Sapporo

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1. The present note is a continuation of a previous paper by the author [3]. O. Szász [13, 14] discussed the following problem concerning the product of two summability methods for sequences: If a sequence $\{s_n\}$ is summable by a regular T_1 method then is the T_2 transform of $\{s_n\}$, where T_2 is a regular sequence-to-sequence method, also summable by the T_1 method to the same sum as before? In what follows we denote $T_1 \cdot T_2$ as the iteration product of these two methods, that is the T_1 transform of the T_2 transform of a sequence. He answered this problem in the affirmative in the several cases. He also gave an example of two regular methods, where T_1 does not imply $T_1 \cdot T_2$. Here we denote "method A implies method B", when any sequence summable A is summable B to the same sum. T. Pati [5], C. T. Rajagopal [7], M. R. Parameswaran [6], M. S. Ramanujan [11, 12], D. Borwein [1] and the author [3] also discussed this problem. M. S. Ramanujan [11] proved the following

Theorem 1. *For a bounded sequence the Abel method A implies the $A \cdot (H^*, \psi)$ method. Here we denote by (H^*, ψ) the regular quasi-Hausdorff method. In the special case when the (H^*, ψ) method gives the circle method of summability (γ, r), the Abel method implies the $A \cdot (\gamma, r)$ method irrespective of whether $\{s_n\}$ is bounded or not.*

The latter part of this theorem was at first established by O. Szász [14]. See for the definition of the quasi-Hausdorff method of summability G. H. Hardy [2] and M. S. Ramanujan [8, 9, 10].

On the other hand M. S. Ramanujan [10] introduced a new method of summability (S^*, ψ) by a modification of the quasi-Hausdorff method. The (S^*, ψ) means of a sequence $\{s_n\}$ are defined by the transformation

$$(1) \qquad s_n^* = \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} s_\nu \int_0^1 (1-t)^\nu t^{n+1} d\psi(t) \quad (n=0, 1, 2, \cdots),$$

where $\psi(t)$ is a function of bounded variation in the closed interval $[0, 1]$. This method is regular if, and only if,

$$(2) \qquad \psi(1) = \psi(1-0)$$

and

$$(3) \qquad \int_{+0}^1 d\psi(t) = 1. \quad \text{(See [10].)}$$

In the special case when, for a given α $(0<\alpha<1)$,