93. A Note on Metric General Connections

By Tominosuke ŌTSUKI

Department of Mathematics, Tokyo Institute of Technology

(Comm. by Z. SUETUNA, M.J.A., Oct. 12, 1962)

In this note, the author will use the notations in [8], [9], [10], [11], [12]. He proved in [9] the following

Theorem A. *Let* $P=P_i^j\partial u_j\otimes du^i$ *and* $G=g_{ij}du^i\otimes du^j$ *be a normal tensor and a non-singular symmetric tensor on an n-dimensional differentiable manifold* $\mathfrak{X}$ *such that* P *is orthogonally related with* G. *Then, there exists a normal general connection* Γ *which satisfies the following conditions*:

(i) $P=\lambda(\Gamma)$, (ii) Γ *is proper, and*

(iii) Γ *is metric with respect to* G.

Furthermore, if we add to them the condition:

(iv) $S_k{}^j{}_h A_i^k = \frac{1}{2} A_l^j (P_{k;h}^l - P_{h;k}^l) A_i^k$,

where A_i^j *are the local components of* A, $S_i{}^j{}_h=\frac{1}{2}(\Gamma_{ih}^j-\Gamma_{ih}^j)$ *and the semi-colon* ";" *denotes the covariant derivatives with respect to Levi-Civita's connection made by* G, *then* Γ *is uniquely determined.*

In this theorem, A is the projection of $T(\mathfrak{X})$ onto the image of P with respect to the direct sum decomposition of $T(\mathfrak{X})$ by means of the image and the kernel of P.

On the other hand, we say a curve $C: u^j=u^j(t)$ in a space $\mathfrak{X}$ with a normal general connection $\Gamma=\partial u_j\otimes(P_i^j d^2u^i+\Gamma_{ih}^j du^i\otimes du^h)$ *is basic*, if its tangent vector at each point is invariant under A. In [12], he proved that *if* Γ *is contravariantly proper, that is*

$$N_k^j \Gamma_{lp}^k A_i^l A_h^p = 0,$$

where $N_i^j=\delta_i^j-A_i^j$, *then we can uniquely parallel translate any A-invariant*[1] *contravariant vector at a point along a basic curve through the point, preserving the A-invariant property and if* Γ *is covariantly proper, that is*

$$A_k^j \Lambda_{lp}^k N_i^l A_h^p = 0,$$

where $\Lambda_{ih}^j=\Gamma_{ih}^j-\partial P_i^j/\partial u^h$, *then the same fact holds good for covariant vectors.*

In [9], a normal general connection Γ was said *proper*, if $N\Gamma=0$,[2] that is

1) We say vectors or tensors are A-invariant, if they are invariant under the homomorphism A of $T(\mathfrak{X})$.

2) See [11], §1.