139. Unique Continuation Theorem of Elliptic Systems of Partial Differential Equations

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1. Let $L$ be a linear partial differential operator defined in a domain $D$ of the $n$-dimensional Euclidean space $\mathbb{R}^n$. For simplicity, we assume that the origin $0$ of $\mathbb{R}^n$ is contained in $D$. Denoting by $x=(x_1, \ldots, x_n)$ a point of $\mathbb{R}^n$, we can write

$$L = L(x, D) = \sum \alpha a_\alpha(x) D^\alpha,$$

where $\alpha$ is a sequence $(\alpha_1, \ldots, \alpha_n)$ of $n$ non-negative integers,

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

and each $a_\alpha(x)$ is a real-valued continuous function in $D$. We use a notation $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Put $r = |x| = (\sum_{i=1}^{n} x_i^2)^{1/2}$.

Friedman [3] proved the following.

Let $u(x)$ be a constant-signed solution, in $D$, of an elliptic differential equation $Lu = 0$ of order $2s$. If

$$\lim_{r \to 0} \frac{D^\alpha u(x)}{r^{|\alpha|}} = 0 \quad \text{for any positive integer } k,$$

where $\alpha$ is an arbitrary sequence with $|\alpha| \leq 2s - 1$, then $u(x)$ vanishes identically in $D$.

And Pederson [4] gave an improvement of this theorem. That is, he proved that, in the above theorem, the assumption (1) can be replaced by the condition that there exists a positive integer $N$ satisfying

$$\lim_{r \to 0} \frac{D^\alpha u}{r^N} = 0$$

for every $\alpha (0 \leq |\alpha| \leq 2s - 1)$ and being dependent on $L$ and independent of $u$.

Now consider an elliptic system of linear partial differential equations

$$\sum_{j=1}^{p} l_{ij} u_j = 0 \quad (i = 1, \ldots, p)$$

in unknown functions $u_1, \ldots, u_p$, where $l_{ij}$ is a linear partial differential operator with variable coefficients continuous in $D$. Carleman [1] proved the following.

In the case when in (2), $p=2, n=2$ and $l_{ij}(i, j=1, 2)$ are of order 1, each solution $u_j$ of (2), satisfying