160. On the Existence of Local Solutions for Some Linear Partial Differential Operators

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(Comm. by K. KUNUGI, M.J.A., Dec. 12, 1962)

1. *Introduction.* A problem of the existence of local solutions is stated in the following; for a linear partial differential operator $P(x, D)$ and some neighbourhood of any fixed point Ω, when we give any element f in $L^2(\Omega)$, we ask whether the equation $P(x, D)u=f$ has at least one solution u in $L^2(\Omega)$ or not, i.e. the Range of $P(x, D)$ equals to $L^2(\Omega)$? By the theorem of the range of linear transformation,[1] we shall prove the inequality of the type $||u||\leqq C||\overline{P}(x, D)u||$[2] for u belonging to $C_0^\infty(\Omega)$ which is the dense subset of the domain of $\overline{P}(x, D)$. B. Malgrange proved the existence of solutions for the operators with constant coefficients.[3] L. Hörmander proved the inequality for the operators with variable coefficients under the conditions of the principal type and that imposed on the commutator $\overline{P}_m(x, D)P_m(x, D)-P_m(x, D)\overline{P}_m(x, D)$.[4),5)] M. Matsumura proved the inequality for the operators whose principal parts,[5] when they are represented products of singular integral operator[6] involving first order differential operators, have factors satisfying the condition of commutator similar to the Hörmander's.[7] As Hörmander's and Matsumura's conditions impose on the principal part of $P(x, D)$, these classes of operators contain, for example, the Laplace and wave operators but not contain the Schrödinger operator of a free particle and heat operator. Now we shall prove the inequality for a class of operators which involves not only the principal type but also the Schrödinger operator of a free particle—but this class does not contain the heat operator, for we consider the operators whose coefficients are real valued. In the case of complex valued coefficients, we shall publish later. The idea is based on Hörmander.[8] This work has been directly inspired by the uniqueness theorem obtained by Kumanogo.[9]

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1) [1] Chap. I Lem. 1.1, [7] Chap. III Th. 6 (§6).
2) $\overline{P}(x, D)$ is adjoint operator of $P(x, D)$.
3) [5].
4) $P_m(x, D)$ is the principal part of $P(x, D)$, cf. [1], [2].
5) The highest order part.
6) In the sense of Calderón and Zygmund.
7) [6].
8) [1], [2], [3].
9) [4].