

8. A Subset Homeomorphic to the Whole Space

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Professor Nagami suggests the problem:¹⁾ Is there a nowhere dense subset which is homeomorphic to the underlying space of an infinite 0-dimensional compact T_1 -group?

The purpose of this paper is to show that this problem is affirmative.

In this paper the dimension means the covering dimension which will be denoted by \dim .

Lemma (L. N. Ivanovskii [3], V. Kuz'minov [4], A. Hulanicki [2], E. Hewitt [1]). *An infinite compact T_1 -group G with $\dim G=0$ is homeomorphic to the generalized Cantor set $\{0, 1\}^\tau$, where τ is the topological weight of G .*

Theorem. *An infinite compact T_1 -group G with $\dim G=0$ has a nowhere dense subset, which is homeomorphic to G .*

Proof. Take a basis whose cardinal is τ , consisting of open and closed sets in G . Well order it as follows, $\{U_1, U_2, \dots, U_\alpha, \dots; \alpha < \mathcal{D}\}$, where \mathcal{D} is the first ordinal with the cardinal τ . For each $\alpha < \mathcal{D}$, φ_α denotes the characteristic function of U_α . We define a mapping Φ of G into $\{0, 1\}^\tau$ by: $\Phi(x) = (\varphi_\alpha(x))_{\alpha < \mathcal{D}}$. Since all the sets U_α are open and closed, Φ is a continuous mapping. Since the sets U_α separate points of G , Φ is one-to-one. Therefore Φ is a homeomorphism, and $\Phi(G)$ is a compact and hence closed subspace of $\{0, 1\}^\tau$. It is impossible that $\Phi(x)=0$ for all but a finite number of α 's at any point x in G . This implies that $\Phi(G)$ has no open subset in $\{0, 1\}^\tau$.

By the lemma, there exists a homeomorphism Ψ of $\{0, 1\}^\tau$ onto G , and $\Psi(\Phi(G))$ is nowhere dense in G .

Corollary 1. *Let G be a locally compact T_1 -group with $\dim G=0$ which has no isolated point. Then G has a nowhere dense subset which is homeomorphic to G .*

This follows immediately from the well-known fact that G has an infinite compact open subgroup.

Corollary 2. *If a locally compact T_1 -group G with $\dim G < \infty$ is not locally connected, then the invariance theorem of domain does not hold in G .*

Since we have the theorem, this is proved by the quite analo-

1) In case of an infinite 0-dimensional compact metric T_1 -group the answer of this problem is affirmative, which was proved in the proof of K. Nagami [5, Theorem 4.2].