4. *On the Existence and the Propagation of Regularity of the Solutions for Partial Differential Equations. II*

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3. Main theorems. Let us resolve L_0 in (1.3) into

$$L_0(t, x, \lambda, \sqrt{-1}\,\eta|\eta|^{-1})=\prod_{i=1}^{k}(\lambda+\lambda_{0,i}^{(1)}(t, x, \eta))\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t, x, \eta)) \tag{3.1}$$

$$(m\geqq k>0,^{6)}\ \eta\neq 0)$$

such that $|\Re e^{7)}\lambda_{0,j}(t, x, \eta)|\geqq\delta>0$ $(j=1,\cdots,m-k)$ with a constant δ. Then, we can write

$$L_0(t, x, \lambda, \sqrt{-1}\,\xi)=\prod_{i=1}^{k}(\lambda+\lambda_{0,i}^{(1)}(t, x, \xi R^{-1})r^{1/m})\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t, x, \xi R^{-1})r^{1/m})$$

with $r=r(\xi)$ defined by (2.1) and R defined by (2.4); see [4].

Theorem 1. *Let L be a differential operator of the form* (1.1) *with bounded measurable coefficients in a neighborhood of the origin, and assume that the coefficients of L_0 are in C^∞.*

Suppose that $\lambda_{0,i}^{(1)}(t,x,\eta)$ $(i=1,\cdots,k)$ are in $C^\infty_{(t,x,\eta)}(\eta\neq 0)$ and distinct, and each $\lambda_i(t, x, \xi)=\lambda_{0,i}^{(1)}(t, x, \xi R^{-1})r^{1/m}$ satisfies the condition

$$\frac{\partial}{\partial t}p_i+\sum_{j=1}^{\nu}\left\{\frac{\partial}{\partial x_j}p_i\frac{\partial}{\partial x_j}q_i-\frac{\partial}{\partial x_j}q_i\frac{\partial}{\partial \xi_j}p_i\right\}=\sigma(H_i)p_i\quad(|\xi|\geqq 1) \tag{3.2}$$

for $p_i=\Re e\,\lambda_i$, $q_i=\Im m\,\lambda_i$ and some $H_i(t)\in C^m_{\mathrm{m}}$. Then, with $\varphi_0=(1+t/2h_0)$ we have a priori inequality

$$\begin{aligned} &n\sum_{i+j=m-1}\int\varphi_0^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right\|^2dt\\ &\leqq C\left\{\int\varphi_0^{-2n}\|Lu\|^2dt+\sum_{i+j=\tau\leqq m-2}n^{2(m-\tau)-1}\int\varphi_0^{-2n}\left\|\frac{\partial^i}{\partial t^i}\Lambda_0^j u\right\|^2dt\right\} \end{aligned} \tag{3.3}$$

$$u\in C_0^\infty(\Omega_{h_0})^{8)}$$

for a sufficiently small fixed h_0 and every $n(\geqq 1)$.

Remark. i) If $P_i\equiv 0$ or $P_i\neq 0$ for any $\xi\neq 0$, the condition (3.2) is always satisfied. ii) Here we do not require the regularity of $\lambda_{0,j}^{(2)}$ $(j=1,\cdots,m-k)$, but in the case when $\lambda_{0,j}^{(2)}$ are in $C^\infty_{(t,x,\eta)}$ $(\eta\neq 0)$ and distinct the uniqueness of the Cauchy problem holds; see [4].

Proof of Theorem 1. Let us write $\prod_{j=1}^{m-k}(\lambda+\lambda_{0,j}^{(2)}(t, x, \eta))=\sum_{j=0}^{m-k}h_{0,j}(t, x, \eta)\lambda^{m-k-j}$ $(h_{0,0}=1)$. Then, from the infinite differentiability of the

6) In the case when we can take $k=0$, L is *hypoelliptic* if the coefficients are in C^∞, and the existence theorem of solutions is easy from Lemma 3 for sufficiently small h. Hence, we may consider only the case $k>0$.

7) For a complex number a, by $\Re e\,a$ we shall denote the real part of a and by $\Im m\,a$ the imaginary part.

8) $\Omega_h=\{(t, x);\ t^2+K(x)^2<h^2\}$.