27. On Conditionally Hypoelliptic Properties of Partially Hypoelliptic Operators

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1. Introduction. Recently L. Gårding and B. Malgrange [2, 3] have introduced the notions of partial hypoellipticity, partial ellipticity and conditional ellipticity. J. Friberg [1] and L. Hörmander [6] proved the fact that the solutions of $P(D)u=0$ is hypoanalytic of type σ in a fixed direction when $P(\zeta)$ is a polynomial of finite type σ in the same direction. J. Friberg also expected in his paper [1] that if $P(D)$ is partially hypoelliptic of type σ in some indepentent variables then the operator $P(D)$ have conditionally hypoelliptic properties in the same variables. (An operator $P(D)$ will be said to have a *conditionally hypoelliptic property of type σ in x'* if any solution $u\in A_{1(x'')}\cap C^\infty$ of $P(D)u=f(f\in A_{1(x)})$ belongs to $A_{\sigma(x)}$. See Def. 2.2.) The object of this note is to give a proof of above fact. The method is based on the idea of Gårding and Malgrange [2]. As the proof is somewhat mazy, details will be published later in the Osaka Mathematical Journal. I should like to thank Prof. M. Nagumo for his kind criticism during the preparation of this paper.

2. Algebraic considerations. Let $P(D)$ be a linear partial differential operator with constant coefficients operating on functions $u(x)$ defined in some open set $\Omega\subset R^m_{x'}\times R^n_{x''}$ $(x=(x',x'')=(x'_1,\cdots,x'_m, x''_1,\cdots,x''_n)\ x'\in R^m, x''\in R^n)$. By α we shall denote a multi-integer $(\alpha^{1'},\cdots,\alpha^{m'},\alpha^{1''},\cdots,\alpha^{n''})$ where $\alpha^{i'}$ and $\alpha^{j''}$ are non-negative integers, the length of α is denoted by $|\alpha|=\alpha^{1'}+\cdots+\alpha^{n''}$. Defining $D_{x'_j}=-\sqrt{-1}\,\partial/\partial x'_j$, $D_{x''_j}=-\sqrt{-1}\,\partial/\partial x''_j$ we set $D^\alpha=D^{\alpha'}_{x'}\cdot D^{\alpha''}_{x''}=D^{\alpha 1'}_{x'_1}\cdots D^{\alpha m'}_{x'_m}\cdot D^{\alpha 1''}_{x''_1}\cdots D^{\alpha n''}_{x''_n}$. By $P(\zeta)$ we mean the characteristic polynomial belonging to $P(D)$, and $V(P)$ denotes the algebraic variety in $C^m\times C^n$ defined by $\{\zeta; P(\zeta)=0\}\subset C^m\times C^n$.

Definition 2.1. The operator $P(D)$ (or $P(\zeta)$) is said to be *partially hypoelliptic of type σ in x'* if the following condition is satisfied.

There exist positive constants C_0 and σ (depending only on P) such that

(2.1) $$|Re\zeta'|\leqq C_0(1+|Im\zeta'|+|\zeta''|)^\sigma \quad (\zeta\in V(P))$$

or equivalently there exist positive constants C'_0 and σ for sufficiently large A

(2.1)′ $$|Re\zeta'|\leqq C'_0(|Im\zeta'|+|\zeta''|)^\sigma \quad (\zeta\in V(P) \text{ and } |Re\zeta'|>A).$$

Remark 1. As in the proof of Lemma 3.9 in Hörmander [5],