18. *On Rings of Analytic Functions on Riemann Surfaces*

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Let R_j be an open Riemann surface and $A(R_j)$ be the ring of all one-valued regular analytic functions on R_j $(j=1,2)$ and σ be a ring isomorphism of $A(R_1)$ onto $A(R_2)$. Since the imaginary unit i is the primitive fourth root of 1, either $i^\sigma=i$ or $-i$. In the former (resp. latter) case, σ is called a *direct* (resp. *indirect*) ring isomorphism. Suppose that there exists a one-to-one transformation S of R_1 onto R_2. If S is directly conformal, then S induces a direct ring isomorphism σ defined by the relation

$$f(p)=f^\sigma(S(p)) \quad (f\in A(R_1),\ p\in R_1).$$

If S is indirectly conformal, then S induces an indirect ring isomorphism σ defined by the relation

$$\overline{f(p)}=f^\sigma(S(p)) \quad (f\in A(R_1),\ p\in R_1).$$

In either case, we say that *σ is induced by S*. The aim of this note is to prove the converse of the above fact.

Theorem. *Any direct (resp. indirect) ring isomorphism of $A(R_1)$ onto $A(R_2)$ is induced by a unique one-to-one direct (resp. indirect) conformal transformation of R_1 onto R_2.*

This fact is first proved by Bers under the assumption that R_1 and R_2 are open plane domains.[1] For arbitrary open Riemann surfaces R_1 and R_2, Rudin proved the above fact under the assumption that the given isomorphism preserves complex constants unchanged.[2] Hence our Theorem, in which no *a priori* assumption on complex constants is made, is a proper generalization of Bers' result and also contains Rudin's result.[3] We divide the proof of our Theorem into several lemmas. Some of them are well known but we include their proofs for the sake of completeness.

1. Ring isomorphism on complex numbers. Let σ be the given ring isomorphism of $A(R_1)$ onto $A(R_2)$ and τ be the inverse of σ. The map τ is also a ring isomorphism of $A(R_2)$ onto $A(R_1)$. We denote by C the complex number field and by C_r the complex rational number field, where a complex number, both of whose real and ima-

1) Bull. Amer. Math. Soc., **54**, 311–315 (1948).

2) Bull. Amer. Math. Soc., **61**, 543 (1955).

3) This problem is suggested by Prof. Bers. If $R_1 \notin O_{AB}$, then our Theorem is easily reduced to Rudin's result. See Proposition 3 in Royden's paper: Seminars on analytic functions, Inst. for advanced study, Princeton, **2**, 273–285 (1958).