15. A Note on Absolute Convergence of Fourier Series

By Kenji YANO

Department of Mathematics, Nara Women's University

(Comm. by Zyoiti SUETUNA, M.J.A., Feb. 12, 1963)

1. Theorems. Let $f(x)$ be integrable in Lebesgue sense in $(0, 2\pi)$, periodic with period 2π, and let

$$f(x) \sim \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

a_0 being, as we may, supposed to be zero. Then, its allied series is

$$\sum_{n=1}^{\infty}(a_n \sin nx - b_n \cos nx).$$

At $x=0$, these series become

$$\sum_{n=1}^{\infty} a_n \quad \text{and} \quad \sum_{n=1}^{\infty}(-b_n)$$

respectively.

In what follows, for the sake of convenience, $\sum_{n=1}^{\infty}$ will be sometimes denoted by $\sum$.

It is well known that f is bounded in $(0, 2\pi)$, and $a_n \geqq 0$ for all n, then $\sum a_n < \infty$. Cf. Paley [1]. But, the proposition $b_n \geqq 0$ for all n does not necessarily imply $\sum b_n < \infty$, unless some additional condition will be assumed concerning the function conjugate to f.

In this paper, we shall prove the following theorems.

THEOREM 1. If $f \in L$, and

$$f_h(0) = \frac{1}{2h}\int_0^h [f(t) + f(-t)]dt \tag{1.1}$$

is bounded for $0 < h < \pi$, then the proposition $a_n \geqq 0$ for all n, or more generally

$$\sum_{n=1}^{\infty}(|a_n| - a_n) < \infty$$

implies

$$\sum_{n=1}^{\infty}|a_n| < \infty.$$

This theorem is clearly trivial when f is odd.

In the case $a_n \geqq 0$, this theorem is due to Szász [2, p. 697].

THEOREM 2. If $f \in L$, and

$$\bar{f}_h(0) = -\frac{1}{\pi}\int_h^{\pi} \frac{f(t) - f(-t)}{2 \tan (t/2)} dt \tag{1.2}$$

is bounded for $0 < h < \pi$, then

$$\sum_{n=1}^{\infty}(|b_n| - b_n) < \infty$$

implies