37. Tauberian Theorems Concerning the Summability Methods of Logarithmic Type

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§ 1. In the recent papers the author proved some theorems concerning the summability methods of logarithmic type. (See [3, 4].) When a sequence $\{s_n\}$ is given we define the method l as follows: If

$$t_0=s_0, \quad t_1=s_1,$$

(1) $$t_n=\frac{1}{\log n}\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right) \quad (n\geq 2)$$

tends to a finite limit s as $n\to\infty$, we say $\{s_n\}$ is summable (l) to s and write $\lim s_n=s(l)$. (See [2] p. 59, p. 87, [5] p. 32.)

On the other hand we define the method L as follows: If

(2) $$f(x)=\frac{-1}{\log(1-x)}\sum_{n=0}^{\infty}\frac{s_n}{n+1}x^{n+1}$$

tends to a finite limit s as $x\to 1$ in the open interval $(0, 1)$, we say that $\{s_n\}$ is summable (L) to s and write $\lim s_n=s(L)$. (See [1].)

When a series $\sum_{n=0}^{\infty}a_n$ is given we define the method l and the method L as before by putting

$$s_n=a_0+a_1+\cdots+a_n \quad (n\geq 0).$$

In the present note we shall prove the following two theorems.

Theorem 1. *If $\sum_{n=0}^{\infty}a_n$ is summable (l) to s, and if*

(3) $$a_n=o\left(\frac{1}{n\log n}\right),$$

then $\sum_{n=0}^{\infty}a_n$ converges to the same value.

Theorem 2. *If $\sum_{n=0}^{\infty}a_n$ is summable (L) to s, and if it satisfies* (3), *then $\sum_{n=0}^{\infty}a_n$ converges to the same value.*

Since the series summable (l) is also summable (L) to the same sum, Theorem 2 includes Theorem 1. (See [3].) However the proof of Theorem 1 seems to be fundamental, we shall prove Theorem 1 first.

§ 2. **Proof of Theorem 1.** From (1) we get

$$s_n-t_n=\frac{1}{\log n}\left\{s_n\log n-\left(s_0+\frac{s_1}{2}+\cdots+\frac{s_n}{n+1}\right)\right\}.$$

Since

$$\log n=1+\frac{1}{2}+\cdots+\frac{1}{n+1}+O(1) \quad \text{as} \quad n\to\infty,$$