33. On A Characterization of Abelian Varieties

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Let G, G' be two group varieties, f_0 a rational homomorphism of G into G', and a a point of G'. Then $f(x)=f_0(x)\cdot a$ for the point x of G, is a rational mapping of G into G'. We shall write more simply $f=f_0\cdot a$. (The same rational mapping f can be also expressed in the form $f=a\cdot f_0'$, where $f_0'=a^{-1}\cdot f_0\cdot a$ is another rational homomorphism of G into G'.) We shall call a rational mapping f which is expressible in the form $f_0\cdot a$ (or $a\cdot f_0'$) a *mapping of type HT* (homomorphism plus translation).

One of the fundamental theorems on abelian varieties asserts that every rational mapping of an abelian variety A into another abelian variety B is a mapping of type HT (cf. [1] Theorem 9). In this theorem, the abelian variety A can be replaced by any group variety G, as was shown by S. Lang [2]. In the present note, we shall prove the converse of this fact in the following sense: Let G, G' be two group varieties. If every rational mapping of G into G' is of type HT, then G' must be an abelian variety.

We shall use the following terminologies and notations. A *homomorphism* of a group variety into a group variety will always mean a rational homomorphism. A *linear group* will always mean a linear algebraic group. A *biregular isomorphism* between group varieties is a group isomorphism defined by a birational mapping which we shall denote by $\cong$. G_a denotes an affine line with the law of composition $z=x+y$, and G_m an affine line, from which the origin is excluded, with the law of composition $z=x\cdot y$. A connected linear group of dimension 1 is isomorphic to G_a or G_m (cf. [1] p. 69). G_a and G_m can be defined over any field k, and their generic points over k are those which have transcendental elements over k as their coordinates. We denote the characteristic of the universal domain by p.

We shall begin with some lemmas.

LEMMA 1. *Every linear group L of dimension $n>0$ has a linear subgroup of dimension 1.*

PROOF. We may assume L as connected. Let L_0 be the Borel subgroup, i.e. the maximal closed solvable connected subgroup, of L, then L/L_0 is a projective variety (cf. Borel [3] Theorem 16.5), so $\dim L_0>0$. As L_0 is solvable, L_0 has a linear subgroup of dimension 1.

LEMMA 2. 1) *Let L_1 and L_2 be connected linear groups of*