49. On Cesàro Summability of Fourier-Laguerre Series

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1. The Fourier-Laguerre expansion corresponding to a function $f(x)\in L(0,\infty)$ is given by

$$f(x)\sim\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(x) \tag{1.1}$$

where

$$\Gamma(\alpha+1)\binom{n+\alpha}{n}a_n=\int_0^{+\infty}e^{-x}x^{\alpha}f(x)L_n^{(\alpha)}(x)dx, \tag{1.2}$$

and $L_n^{(\alpha)}$ denotes the Laguerre polynomial of order α.

At the point $X=0$

$$\sum_{n=0}^{\infty}a_nL_n^{(\alpha)}(0)=\frac{1}{\Gamma(\alpha+1)}\sum_{n=0}^{\infty}\int_0^{\infty}e^{-t}t^{\alpha}f(t)L_n^{(\alpha)}(t)dt. \tag{1.3}$$

Denoting the Cesàro means of order k of the series (1.1) at the point $X=0$ by $\sigma_n^k(0)$, we easily have

$$\sigma_n^k(0)=\{A_n^{(k)}\Gamma(\alpha+1)\}^{-1}\int_0^{\infty}e^{-t}t^{\alpha}f(t)L_n^{(\alpha+k+1)}(t)dt. \tag{1.4}$$

Szegö [1] has studied the (C,k) summability of Laguerre series corresponding to a continuous function for $k>\alpha+\frac{1}{2}$.

In the present paper I prove the following more general theorem:–

Theorem. If $f(x)$ be integrable in $(0,\infty)$ and if it satisfies the following conditions

$$\int_1^{\infty}e^{-\frac{x}{2}}x^{\alpha-k-\frac{1}{3}}|f(x)|dx<\infty,\ \text{and} \tag{1.5}$$

$$\int_0^t|f(t)|dt=o(t), \tag{1.6}$$

then the Laguerre series of $f(x)$ is (C,k) summable at $x=0$ with the sum of $f(0)$ provided that $k>\alpha+\frac{1}{2}$.

2. We shall take help of the following lemmas in the proof of the theorem:—

Lemma 1 (Szegö [2], p. 172). Let α be arbitrary and real, C and ω fixed positive constants, and let $n\to\infty$. Then

$$L_n^{(\alpha)}(x)=\begin{cases}x^{-\frac{\alpha}{2}-\frac{1}{4}}0(n^{\frac{\alpha}{2}-\frac{1}{4}}), & \text{if, } \frac{c}{n}\le x\le\omega,\\ 0(n^{\alpha}) & \text{if, } 0\le x\le\frac{c}{n},\end{cases} \tag{2.1}$$

Lemma 2 (Szegö [2], p. 235). Let α and λ be arbitrary and real