47. Some Properties of Completely Normal and Collectionwise Normal Spaces

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(Comm. by Kinjirô KUNUGI, M.J.A., April 12, 1963)

1. In our previous note [4] we have proved the following theorem.

Theorem 1. *If for any locally finite family* $\{X_\alpha \mid \alpha\in\Omega\}$ *of closed subsets of a topological space* R *there exists a locally finite covering* $\{H_\alpha \mid \alpha\in\Omega\}$ *of closed subsets of* R *satisfying one of the following equivalent conditions* (A), (B), *and* (C), *then* R *is completely normal and collectionwise normal:*

(A) $H_\alpha\frown H_\beta\frown(X_\alpha\smile X_\beta)=X_\alpha\frown X_\beta,\qquad (\alpha,\beta\in\Omega)$

(B) $H_\alpha\frown H_\beta=X_\alpha\frown X_\beta,\qquad (\alpha,\beta\in\Omega)$

(C) $H_\alpha\frown(\smile_{\gamma\in\Omega}X_\gamma)=X_\alpha\qquad (\alpha\in\Omega).$

In this paper, in connection with the above theorem we shall establish necessary and sufficient conditions for topological spaces to be completely normal and collectionwise normal (see Theorems 3 and 4).

2. We shall first prove the following theorem.

Theorem 2. *Let* R *be a completely normal space. If* $\{X_\alpha \mid \alpha\in\Omega\}$ *is a family of closed subsets of* R *and* $\{U_\alpha \mid \alpha\in\Omega\}$ *is a locally finite family of open subsets of* R *such that* $X_\alpha\subset U_\alpha$ *for each* $\alpha\in\Omega$, *then there exists a locally finite closed covering* $\{H_\alpha \mid \alpha\in\Omega\}$ *of* R *satisfying one of the equivalent conditions* (A), (B), *and* (C) *of Theorem 1.*

Proof. If Ω is a finite set, this theorem has already been established (see [4], Theorem 1), so we assume that Ω is infinite. Now, we shall construct a family $\{H_\alpha \mid \alpha\in\Omega\}$ of closed subsets of R satisfying the condition (C) of Theorem 1 by transfinite induction.

Let η be a limit ordinal number such that $\Omega=\{\alpha \mid \alpha<\eta\}$ and let us put $U=\smile_{\alpha\in\Omega}U_\alpha$, $U'_\mu=\smile_{\alpha>\mu}U_\alpha$, $X=\smile_{\alpha\in\Omega}X_\alpha$, $X'_\mu=\smile_{\alpha>\mu}X_\alpha$. Let ν be an ordinal number such that $\nu<\eta$. We assume that to every $\mu<\nu$ there exist two closed subsets F_μ, F'_μ of U satisfying the following conditions:

$$(P_\mu)\quad\begin{cases}(1)\ \ U_\gamma\supset F_\gamma,\ \ (\gamma\leqq\mu);\ \ U'_\mu\supset F'_\mu,\\ (2)\ \ (\smile_{\gamma\leqq\mu}F_\gamma)\smile F'_\mu=U,\\ (3)\ \ F_\gamma\frown X=X_\gamma\qquad(\gamma\leqq\mu),\\ (4)\ \ F'_\mu\frown X=X'_\mu.\end{cases}$$

Then we shall construct two closed subsets F_ν, F'_ν of U satisfying the condition (P_ν).

We shall first show that the relation $\frown_{\mu<\nu}\smile_{\alpha>\mu}U_\alpha=\smile_{\mu\geqq\nu}U_\mu$ holds. It is evident that $\frown_{\mu<\nu}\smile_{\alpha>\mu}U_\alpha\supset\smile_{\mu\geqq\nu}U_\mu$. So conversely, let $x(\in U)$ be any