67. *Semigroups of Positive Integer Vectors*[1]

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1. Consider the set I of all the vectors $(x_1,\cdots,x_i,\cdots,0,\cdots)$ with countable components of non-negative integers where a finite number of x_i's are positive and the remaining are 0, but all are not 0. The addition is defined as follows:

$$(x_1,\cdots,x_i,\cdots)+(y_1,\cdots,y_i,\cdots)=(x_1+y_1,\cdots,x_i+y_i,\cdots)$$

in which x_i+y_i is the usual addition of integers. Then I forms a semigroup with respect to the addition. We want to determine all the subsemigroups of I from the standpoint of the bases, and by using these results, we can determine the subsemigroups of the multiplicative semigroup of some positive integer vectors. The detailed proof will be given in another paper.

When n is a positive integer and $a\in I$, na denotes $\underbrace{a+\cdots+a}_{n}\in I$.

For convenience, I is considered as a subsemigroup of the module R of all vectors whose components are rational numbers and all are 0 except a finite number of x_i's where scalar-multiplying is regarded as an operator, that is, if λ is a rational number and $a\in I$, $\lambda a =\lambda(a_1,\cdots,a_i,\cdots)=(\lambda a_1,\cdots,\lambda a_i,\cdots)$ where λa_i is the usual product of λ and a_i.

2. Let M be a subsemigroup of I. If M can be embedded into the subsemigroup $I_k=\{x\in I\,|\,x=(x_1,\cdots,x_k,0\cdots),\ x_i=0,\ i>k\}$ and never into $I_{k'}(k'<k)$, then the dimension of M is said to be k and denoted by $\dim. M=k$ or M is called a k-dimensional subsemigroup of I, or, simply, k-dimensional semigroup. If there is no finite k, the dimension of M is said to be infinite. If M is k-dimensional, every element x of M can be expressed as $x=(x_1,\cdots,x_k)$ without loss of generality. A subset B of a finite or infinite dimensional M is called a generator system if each element a of M is of the form $a=\lambda_1b_1+\cdots+\lambda_mb_m$ where λ_i are positive integers and $b_i\in B$ and m is not fixed. If B is minimal generator system of M in the sense of the inclusion relation, then B is called a basis of M. If B consists of finite elements, M is said to have a finite basis, and the number of elements of B is called the basis-order of M.

Lemma 1. *A subsemigroup M of I has a unique basis.*

A finite number of elements $b_1,\cdots,b_m$ in M are said to be linearly

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