66. A Note on Rings of which any One-sided Quotient Rings are Two-sided

By Yuzo UTUMI
The University of Rochester

(Comm. by Kenjiro SHODA, M.J.A., May 11, 1963)

0. A ring S is called a J-ring if the left and right singular ideals vanish. For any J-ring S we may construct the maximal left quotient ring $\bar{S}_1$ and the maximal right quotient ring $\bar{S}_r$.

A left ideal A of a ring S is called closed if there exists a left ideal B such that A is maximal among left ideals disjoint to B. If S is a J-ring it is known that the set of closed left ideals forms a complete complemented modular lattice $L(S)$. Similarly we define closed right ideals, and denote the lattice of closed right ideals by $R(S)$.

We shall show the following two theorems:

Theorem 1. Let S be a J-ring, and suppose that both $L(S)$ and $R(S)$ are atomic. Then the following conditions are equivalent:

(A_1) The right annihilator of an atom of $L(S)$ is a dual atom of $R(S)$.

(B_1) For any atom A of $L(S)$ there exists an atom B of $R(S)$ such that $A\cap B\neq 0$.

(C_1) The right annihilator of the sum of atoms of $R(S)$ is zero.

Theorem 2. Let S be a J-ring. Suppose that both $L(S)$ and $R(S)$ are finite dimensional. Then $\bar{S}_1=\bar{S}_r$ if and only if S satisfies (A_1) and its right-left symmetry (A_r).

Similar results have been obtained by R. E. Johnson too in a completely different way.

1. We denote by $1(*)$ ($r(*)$ resp.) the left (right resp.) annihilator of $*$.

Proof of Theorem 1. $(A_1)\Rightarrow(B_1)$. Let X be an atom of $L(S)$, and let $0\neq x\in X$. Then $r(X)$ is a dual atom of $R(S)$ by assumption, and so $r(x)=r(X)$ since any annihilator right ideal is closed. Let A and B be nonzero right ideals contained in xS. Then we may suppose that $A=xA'$ and $B=xB'$ for some right ideals A' and B' which contain $r(x)$ properly. Now A' and B' are large, and so is $A'\cap B'$. Hence $0\neq x(A'\cap B')\subset A\cap B$. This shows that xS is uniform, therefore the closure of xS is an atom of $R(S)$. Since the closure contains x, this proves (B_1).

$(B_1)\Rightarrow(C_1)$. Let P be the sum of atoms of $R(S)$. Then $P\cap X$